Integral vectors orthogonal to a large set of bounded weight (0, 1)-vectors

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Let us consider a linear subspace of codimension one generated by $(0,1)$-vectors. It corresponds to an integral linear form $c_1x_1 + \ldots + c_nx_n$ for some integer c_1,..., c_n with the greatest common divisor $\mathrm{GCD}(c_1, \ldots, c_n) = 1$. We shall look up the greatest coefficient $g_1(w, n)$ which appears in such linear forms vanishing on a set of bounded weight $(0, 1)$-vectors whose weights are bounded by w. The weight is the sum of all coordinates. A bound means the upper bound. $g_1(w, n) \leq g_1(n, n) \leq 2^{-n}(\sqrt{n+1})^{n+1}$, see [1]. The bound is almost tight in case $w = n$, see [2, 3]. For small weights w there are much better bounds $g_1(w, n) \leq (\sqrt{w})^n$. The last inequality coincides with Hadamard's inequality for determinants. In particular, $g_1(3, n) < (1.73)^n$.

Proposition. *There exist two positive numbers α and β such that the first inequality $g_1(3, n) > \alpha(1.46)^n(1 + o(n))$ holds for all odd n, and the second inequality $g_1(3, n) > \beta(1.46)^n(1 + o(n))$ holds for all even n.*

Proof. Let us consider the infinite system of the linear equations with coefficients from the set $\{0, 1\}$ with at most three nonzero ones in each row.

$$\begin{cases} c_2 + c_1 &= 0 \\ c_3 + c_2 &= 0 \\ c_{2k+2} + c_{2k-1} + c_{2k+1} &= 0 \\ c_{2k+3} + c_{2k} + c_{2k+2} &= 0 \end{cases}$$

The system has a unique solution up to an arbitrary value of c_1. Moreover, $c_n = c_{n-2} + 2c_{n-4} + c_{n-6}$. A calculation performed with Maple have shown all roots of the polynomial $x^6 - x^4 - 2x^2 - 1$ are approximately equal to $\{\pm 1.465571232, \pm 0.2327856159 \pm i0.7925519930\}$. Thus, for some A, B there holds $c_n = (A + (-1)^n B)(1.465571232\ldots)^n(1 + o(n))$.

Remark. Contrariwise, $g_1(2,n)=1$.

The situation varies depending on the cardinality of the set of $(0,1)$-vectors where the linear form vanishes. The set can be large [4]. Let us denote ν_2 the quadratic Veronese map $\nu_2(x_1,\dots,x_n)=(x_1^2,x_1x_2,\dots,x_{n-1}x_n,x_n^2)$. For $w\geq 2$ let us denote $g_2(w,n)$ the greatest coefficient in case of linear forms vanishing on each of $\frac{n(n-1)}{2}$ $(0,1)$-vectors whose ν_2-images are linearly independent. Of course, these vectors themselves are linearly dependent. Obviously, $g_2(2,n)=1$.

Proposition. $g_2(3,n)=1$.

Sketch of the proof. For $n=2$ all weights are bounded by two. Thus, $g_2(3,2)=g_2(2,2)=1$.

Step. Assume $g_2(3,n)=1$ holds for some unspecified value of n. Let us consider a linear form $f(\mathbf{x})=c_1x_1+\ldots+c_{n+1}x_{n+1}$ vanishing on the large set S of $(0,1)$-vectors whose weights are bounded by three.

If there is $\mathbf{v}\in S$ such that $v_i=1$ and for all $j\neq i$ $v_j=0$ then $c_i=0$. The dimension can be reduced by means of the projection which forgets the i-th coordinate. Thus, $g_2(3,n+1)=g_2(3,n)=1$.

Else if each vector in S has exactly two nonzero coordinates then we have a system of equalities $c_i=\pm c_j$ with a unique solution up to some integer factor. Thus, all normalized coefficients c_i belong to the set $\{0,\pm 1\}$.

Else each vector in S has either two or three nonzero coordinates. Let us count unordered pairs of the nonzero coordinates for each vector. The total number of unordered pairs of the nonzero coordinates in the set S is greater than the cardinality of S. There is $(1+\varepsilon)$-fold cover on average, where $0<\varepsilon\leq 2$. Thus, there are two indices i and j such that two vectors $\mathbf{u},\mathbf{v}\in S$ have $u_i=u_j=v_i=v_j=1$. There are two cases. If both vectors $\mathbf{u}$ and $\mathbf{v}$ have equal weights then there are two another indices $k\neq\ell$ such that $u_k=v_\ell=1$ and $c_k=c_\ell$. The substitution leads to reduction of the dimension. If $\mathbf{u}$ has only two nonzero coordinates then the third nonzero coordinate $v_k=1$ corresponds with zero value $c_k=0$. Again the dimension can be reduced by means of the projection which forgets the k-th coordinate. In any case $g_2(3,n+1)=g_2(3,n)=1$.

Conjecture. $g_2(w,n)\ll g_1(w,n)$.

Remark. Such bounds could be useful for the knapsack [5] and related problems [6, 7, 8]. In short, there are the more ways to load the knapsack, the easier to calculate optimum loading by means of dynamic programming [9].

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