### Some applications of finite-support products of Jensen's minimal $\Delta_3^1$ forcing

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Jensen 1970 defined a (forcing)  $J \in L$  such that any J-generic real a:

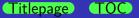
- does not belong to L;
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It is the most elementary example of a Goedel-nonconstructible definable real !









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- 2 a definable Vitali-equivalence class w/o a definable element;



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- I a definable Groszek-Laver pair of Vitali classes;
- a Π<sub>2</sub><sup>1</sup> set P ⊆ ℝ × ℝ, such that:
  1) P is non-uniformizable by ROD sets, and
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- in a choiceless model: a countable sequence of Vitali classes whose union is uncountable.



#### Jensen's basic result

- **2** Countable product of Jensen's forcing
- Oriation: Vitali-invariant forcing
- Groszek Laver pairs of Vitali classes
- **5** Infinite products of large trees

### 6 Final remarks



# Section 1. Jensen's basic result









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#### Theorem (Jensen 1970)

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Jensen's forcing J consists of **perfect sets**  $X \subseteq \mathbb{R}$ , a subset of the Sacks forcing

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# Section 2. Countable product of Jensen's forcing

#### Countable definable set w/o definable elements **TOC Back**

 $J^{\omega}$  is the finite-support countable product of Jensen's forcing J

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Here  $\Pi_2^1$  is the best possible since any non-empty  $\Sigma_2^1$  set of reals surely contains a definable element.



## Section 3. Variation: Vitali-invariant forcing

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**W** is **Vitali-invariant**, that is, is invariant under rational shifts.





Thus there are two methods of getting a countable  $\Pi_2^1$  set  $X \subseteq \mathbb{R}$  containing no definable elements:

(1)  $X = \{x_n : n < \omega\}$ , where  $x_n$  are independently **J**-generic reals added by the finite-support product  $J^{\omega}$  of Jensen's forcing J;

(2) X is the Vitali class of a  $\mathbf{K}$ -generic real x.



# Section 4. Groszek – Laver pairs of Vitali classes





A Groszek – Laver pair is any pair of sets  $X \neq Y \subseteq \mathbb{R}$  inseparable by an OD (ordinal-definable) set, that is, if  $S \subseteq \mathbb{R}$  is OD then

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The sets X, Y is this example are obviously uncountable. Is there an OD Groszek – Laver pair of **countable** sets in  $\mathbb{R}$ ?





## Theorem 3 (Golshani + K + Lyubetsky, MLQ 2016)

There is a special Vitali-connected version of the forcing product  $(\mathbf{K} \times \mathbf{K})$  in L, which adds a pair of reals x, y such that

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The Vitali-connected product consists of all pairs  $\langle X, Y \rangle$  of sets  $X, Y \in \mathbf{K}$  such that  $X + \mathbb{Q} = Y + \mathbb{Q}$ .

An old idea of Harrington – Marker – Shelah, 1990.



# Section 5. Infinite products of large trees









#### It is consistent with **ZFC** that there is a set $W \subseteq \mathbb{R} \times \mathbb{R}$ such that

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## Theorem 4 (K and Lyubetsky, APAL, 2016, 167, 3)

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The forcing is essentially a finite-support product  $\prod_{\alpha < \omega_1} \mathbf{K}_{\alpha}$ , where each  $\mathbf{K}_{\alpha}$  is a clone of the forcing notion  $\mathbf{K}$ .

#### Theorem 4 (K and Lyubetsky)

It is consistent with **ZF** (no axiom of choice!) that there is a countable sequence  $\{X_n\}_{n<\omega}$  of Vitali classes  $X_n \subseteq \mathbb{R}$ , such the union  $X = \bigcup_n X_n$  is not countable.



# Section 6. Final remarks





## Theorem (still work in progress, with Ali Enayat)

There is a generic model in which

• every analytically definable non-empty set of reals contains an analytically definable element;

• there is no projective wellordering of the reals.









## Problem (countable definable sets not of reals)

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#### Problem (countable definable sets not of reals)

Is it true in the Solovay model that every countable definable non-empty set X of any kind contains a definable element?

Yes if X is a **set of reals**.

The most elementary open case: X is a ctble set of sets of reals.



# The speaker thanks everybody for patience

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#### Definition

A set  $X \subseteq \mathbb{R}$  is Vitali-large if the Vitali equivalence restricted to X has no Borel transversal.

A **transversal** for Vitali restricted to X is a set  $Y \subseteq X$  which meets any Vitali class in X in exactly one point.

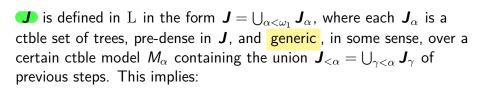
#### Example

The whole set  $\mathbb{R}$  is Vitali-large.



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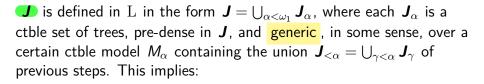
J is defined in L in the form  $J = \bigcup_{\alpha < \omega_1} J_\alpha$ , where each  $J_\alpha$  is a ctble set of trees, pre-dense in J, and generic, in some sense, over a certain ctble model  $M_\alpha$  containing the union  $J_{<\alpha} = \bigcup_{\gamma < \alpha} J_\gamma$  of previous steps. This implies:



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(\*) If  $n < \omega$ , a set  $D \in M_{\alpha}$ ,  $D \subseteq (\mathbf{J}_{<\alpha})^n$  is dense in  $(\mathbf{J}_{<\alpha})^n$ , and trees  $T_1, \ldots, T_n \in \mathbf{J}_{\alpha}$  are 2wise different, then there is a finite subset  $D' \subseteq D$  such that  $T_1 \times \ldots \times T_n \subseteq \bigcup D'$ .

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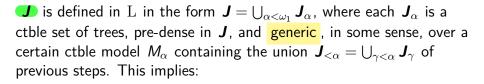


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The main technical problem for **Theorem 1** was to adapt (\*) to the case of the infinite product  $(J_{<\alpha})^{\omega}$ .

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Forcing is a tool to define set theoretic universes.

A universe is a structure of sets which satisfies the axioms of ZFC, Zermelo – Fraenkel axiomatic system, with the axiom of choice.

The minimal universe is the universe L of Goedel-constructible sets.

More universes can be obtained as  $\frac{1}{1}$  extensions of L by forcing, called forcing extensions.

The properties of a forcing extension depend on the choice of a partially-ordered set, called **a forcing notion**, or just **a forcing**.







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- OEP holds for all definable sets X in the Goedel constructible universe L — because of the canonical OD wellordering of L.

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- **3 DEP** fails in many set theoretic universes for instance for the set  $X = \mathbb{R} \setminus L$  of all non-constructible reals.

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- **3 DEP** fails in many set theoretic universes for instance for the set  $X = \mathbb{R} \setminus L$  of all non-constructible reals.

However until recently all known counterexamples to **DEP** have been rather large sets, definitely uncountable. Therefore one can ask: is there a countable counterexample to **DEP** in some universe? \*\*