Minimal collapse maps at arbitrary projective level

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   1. $\boldsymbol{a}$ is (coded by) a lightface Π^1_n real singleton, but
   2. every Σ^1_n real is constructible.

Definition (Cohen-style collapse forcing)

The forcing ${\omega_1}^{<\omega}$ consists of all **strings** (finite sequences) of ordinals $\alpha < \omega_1$.

The forcing ${\omega_1}^{<\omega}$ naturally adjoins a map $\boldsymbol{a} : \omega \xrightarrow{\text{onto}} \omega_1$.

Definition (Minimal cofinal map forcing, Prikry + folclore)

The forcing $\mathbb{P}$ consists of all **trees** $T \subseteq {\omega_1}^{<\omega}$ such that

1. every node of T has a branching node above it;
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Both $\mathbb{P}$ and $\mathbb{P}_{\text{Laver}}$ naturally adjoin a cofinal map $\boldsymbol{a} : \omega \to \omega_1^{\mathbf{V}}$, but such a map $\boldsymbol{a}$ is not definable in $\mathbf{V}[\boldsymbol{a}]$ since the forcing notions $\mathbb{P}$ and $\mathbb{P}_{\text{Laver}}$ are too homogeneous.

Definition (Uri Abraham forcing, in **L**)

In **L**, the forcing $\mathbb{U}$ is a subset $\mathbb{U} = \bigcup_{\xi<\omega_2} \mathbb{U}_\xi \subseteq \mathbb{P}$, such that

1. each $\mathbb{U}_\xi \subseteq \mathbb{P}$ is a set of cardinality $\aleph_1$;
2. $\mathbb{U}$ adds a single generic map, so $\mathbb{U}$ is **very** non-homogeneous;
3. "being $\mathbb{U}$-generic" is Π^1_2.

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- The forcing $\mathbb{U}$ adjoins a cofinal map $\boldsymbol{a} : \omega \to \omega_1$ to **L**, and $\boldsymbol{a}$ is a Π^1_2-singleton in the extension.
- The single generic object construction goes back to Jensen 1970 minimal-Π^1_2-singleton forcing.

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Definition ($\mathit{\Pi}^1_n$-singleton cofinal map forcing)

Let $n \ge 3$. In **L**, we define $\mathbb{U}_n$ using a Δ^1_n path through $\mathfrak{P}$, generic so it meets all dense subsets of $\mathfrak{P}$ of boldface class $\mathbf{\Sigma}^1_{n-1}$.

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The genericity condition makes the forcing properties of $\mathbb{U}_n$ to be very close to those of the whole **homogeneous** forcing notion $\mathbb{P}$ up to the nth level of the projective hierarchy.
In particular $\mathbb{U}_n$ forces all lightface Σ^1_n reals to be constructible.

Problem

In the context of the Namba forcing, define a generic extension $\mathbf{L}[\boldsymbol{a}]$ of $\mathbf{L}$ by a cofinal map $\boldsymbol{a} : \omega \to \omega_2^{\mathbf{L}}$, such that $\omega_1^{\mathbf{L}}$ is not collapsed and $\boldsymbol{a}$ is definable in $\mathbf{L}[\boldsymbol{a}]$.

The speaker thanks **the organizers** for support

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