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Lebesgue measure and gambling

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Abstract. Lebesgue measure of point sets is characterized in terms of the existence of various strategies in a certain coin-flipping game. 'Rational' and 'discrete' modifications of this game are investigated. We prove that if one of the players has a winning strategy in a game of this type depending on a given set $P \subseteq [0,1]$, then this set is measurable.

Bibliography: 11 titles.

Introduction

A typical coin-flipping game has a gambler betting on whether a coin flip will turn up heads or tails (so that the probabilities of heads and tails are both $1/2$), and the gambler returns his bet and is additionally paid the amount of his bet when he correctly predicts the outcome, or otherwise he loses his bet. Martingale theory guarantees that there is no betting strategy for the gambler that (in finite time) will turn this game into a favourable one [1]. That is to say, after n rounds of this game the gambler's expected profit is zero, regardless of the betting strategy employed.

The game we consider in this paper involves two main modifications of this elementary game. First, we give the casino a more active role by allowing it to pick heads or tails for each coin flip. And, on making this choice the casino will know whether the gambler is betting on heads or tails. To compensate for this advantage, we allow the game to continue for infinitely many steps and require the casino to produce a sequence of flips that belongs to a prescribed pay-off set.

We consider the space $\mathbb{D} = \{-1,1\}^{\mathbb{N}}$ equipped with a countable product of the Bernoullian probability measure on the set $\{-1,1\}$ that gives value $1/2$ to either point. It is well known that then the Lebesgue measure of any set $P \subseteq \mathbb{D}$ is equal to the probability that a coin flipped independently and infinitely often will produce a sequence in P. (We identify heads with 1 and tails with -1.) Thus, we have replaced the local probabilistic event that a fair coin comes up heads or tails by the global probabilistic event that a sequence of flips belongs to P.

In § 1 we define the modified game and related notions. Two key strategies of players in this game are considered in § 2. § 3 analyses the case when the pay-off set P has null measure. We study the positive-measure case from the gambler's point of view in § 4 and from the point of view of the casino in § 5. Some estimates

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of the outer and the inner Lebesgue measures of a given set P, in terms of the existence of certain strategies in this game, are given in those sections. These estimates belong to the main results of this paper. They can be interesting even for knowingly measurable sets P, when there is no need to distinguish between the outer and the inner measures.

§§ 6 and 7 contain two modifications of our game. In the first we require the bets by the gambler to be rational (as opposed to an arbitrary real number). It is used to give another proof (from the axiom of determinacy) that all sets are Lebesgue measurable. Therefore, the game under consideration is connected with the set-theoretic question of Lebesgue measurability/non-measurability of point sets. The second, 'discrete' modification, where we require the bets to be integer multiples of a fixed real number, is closer to an actual casino-style game (where bets are generally restricted to integer dollar amounts). We shall see that this modification gives the casino a definite advantage.

We use $\mathbb{R}$ to denote real numbers, $\mathbb{N}$ for natural numbers (including 0), and $\mathbb{D} = \{-1,1\}^{\mathbb{N}}$ to denote the Cantor set of all infinite dyadic sequences with terms equal to -1 and 1, with the usual product topology. We denote by $\mathbf{S} = \{-1,1\}^{<\mathbb{N}} = \bigcup_{n\in\mathbb{N}}\{-1,1\}^n$ the set of all finite sequences of -1 and 1, and $s^\frown t$, for $s,t \in \mathbf{S}$, is the concatenation of s and t, that is, the terms of t follow the terms of s (with preservation of order) within either group. Further, $s \subset t$ means that s is a proper initial segment of the sequence t, while $s \subseteq t$ allows s and t to coincide. The *length* $\operatorname{lh} s$ of a finite sequence s is equal to the unique number $n \in \mathbb{N}$ such that $s \in \{-1,1\}^n$. If a sequence $\vec{x}$ is either infinite or finite of length not less than n, then $\vec{x} \restriction n = \langle x_0, x_1, \dots, x_{n-1}\rangle$ denotes the initial segment of $\vec{x}$ of length n. The basis of clopen sets of $\mathbb{D}$ consists of Baire intervals $\mathbb{D}_s = \{\vec{x} \in \mathbb{D} : s \subset \vec{x}\}$; thus $\mathbb{D}_s$ contains all infinite extensions $\vec{x} \in \mathbb{D}$ of a finite sequence $s \in \mathbf{S}$. Lebesgue measure on $\mathbb{D}$ (the homogeneous probability product measure) is denoted by λ.

§ 1. A modification of the standard coin flipping game

The game described here emerged as the result of analysis of another coin flipping game, introduced in [2] as a basic tool of definition of the notion of an infinite sequence random in the sense of the computability theory. (See a popular exposition of related issues in [3].)

A typical coin flipping game has a gambler betting on whether a coin flip will turn up heads or tails. If the gambler plays this game once for each natural number $n \in \mathbb{N}$, the coin flips produce a sequence $\langle p_n\rangle_{n\in\mathbb{N}}$, of numbers $p_n = 1$ (heads) and $p_n = -1$ (tails), that is, an element of the set $\mathbb{D}$. Our modification introduces a second player who is given the option of selecting the sequence $\langle p_n\rangle_{n\in\mathbb{N}}$, digit by digit, with the requirement that the sequence belongs to a given pay-off set $P \subseteq \mathbb{D}$. Thus the game has two players, identified as Gambler and Casino. At the start of the game, a non-empty pay-off set $P \subseteq \mathbb{D}$ is fixed. We assume that Gambler begins with the initial balance of $B_0 = 1$ dollars. The game has one turn for each natural n and each turn is played so that Gambler places a bet b_n (a real number of absolute value not exceeding B_n, his current balance), and then, seeing this bet, Casino plays a digit, $p_n = +1$ or $p_n = -1$ and Gambler's balance is updated to

$$B_{n+1} = B_n + p_n b_n.$$

Hence, a negative bet by Gambler is a stake that Casino will play $p_n = -1$, and a positive bet by Gambler is a stake that Casino will play $p_n = 1$, since these two situations result in an increase in Gambler's balance. During the course of a run in the game, Casino produces an infinite sequence of flips $\vec{p} = \langle p_n \rangle_{n\in\mathbb{N}} \in \mathbb{D}$ ($p_n = \pm 1$ is the nth digit played by Casino), and Gambler produces a sequence of bets $\vec{b} = \langle b_n \rangle_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}}[-B_n, B_n] \subset \mathbb{R}^{\mathbb{N}}$, where B_n is Gambler's balance at the start of round n. The actual order in which these plays are made is:

$$b_0, p_0, b_1, p_1, \dots, b_n, p_n, \dots,$$

and both players have total knowledge of all things played earlier. In particular, when Casino plays p_n, it knows the value of b_n. To make the game non-trivial we require that Casino produces a sequence $\vec{p} = \langle p_n \rangle_{n\in\mathbb{N}}$ that belongs to a given pay-off set P. We shall refer to this game as $\Gamma(P)$.

We do not directly define the result of a run in this game (that is, the decision who is the winner after the whole infinite sequence of moves is made). This is because the results on games of the form $\Gamma(P)$ that we obtain all have the following general form: one of the two players has a strategy which will guarantee some inequality between either the limit or the supremum of Gambler's balances and the reciprocal of the measure $\lambda(P)$ of the pay-off set P. However, no single form of such an inequality will capture all the results related to this game simultaneously. Hence there is no definition fixed once and for all of who the winner should be. Yet the general goal of Gambler is to increase the balances B_n (or to force Casino to play a sequence $\vec{p} = \langle p_n \rangle_{n\in\mathbb{N}}$ outside P). Accordingly, the general aim of Casino is to decrease B_n (and to stay inside P).

Consider now a version of the game of the form $\Gamma(P)$ which does contain the definition of a winner. Suppose that a non-negative real number $H \geqslant 0$ is fixed at the outset along with the set P; here H can also be equal to ∞. In most cases H will be close to $1/\lambda(P)$. The game $\Gamma(P, H)$ is played in an identical manner to $\Gamma(P)$, but we say in addition that Gambler wins a run of $\Gamma(P, H)$ if one of the following two conditions is satisfied:

1) $\vec{p} \notin P$;
2) $\vec{p} \in P$, $\forall\, n \in \mathbb{N}$ ($|b_n| \leqslant B_n$), and in addition
   (a) $\sup_{n\in\mathbb{N}} B_n > H$ in the case when $H < \infty$,
   (b) $\sup_{n\in\mathbb{N}} B_n = \infty$ in the case when $H = \infty$.

If we say that Casino 'cheats' whenever $\vec{p} \notin P$, and Gambler 'cheats' by betting more than his current balance, then we have declared Gambler the winner, in particular, if either Casino or both players cheat. Clearly, neither player can force the other to cheat, so we shall assume throughout the paper that clause 2) always holds. And in this case clauses 2), (a) and 2), (b) define our notion of winner for H finite or infinite, respectively.

Intuitively, Gambler is given a balance of 1 dollar and is betting on an event $\vec{p} \in P$ of probability $\lambda(P)$ (when P is Lebesgue measurable). Thus, we might expect that Gambler can increase his balance to the value $1/\lambda(P)$. On the other hand, if P has null measure but is dense, it is not quite clear whether Gambler can earn an infinite amount of money in this case. Still we shall see that this is possible.

We now define the notion of a strategy for the two players. Generally, a strategy in games of this type is a rule that tells the player (on the basis of the previous turns) what to play on each turn. For instance, when it is Gambler's turn to play, the previous turns (if it is not the initial turn) have produced:

- the current balance $B_n = B_0 + \sum_{i=0}^{n-1} p_i b_i \in \mathbb{R} \cap [0, \infty)$;
- a sequence $\vec{b} \restriction n = \langle b_0, b_1, \dots, b_{n-1} \rangle \in \mathbb{R}^n$ of bets by Gambler;
- a sequence of digits played by Casino, $\vec{p} \restriction n = \langle p_0, \dots, p_{n-1} \rangle \in \{-1, 1\}^n$;

and Gambler's strategy must define the next bet $b_n \in [-B_n, B_n]$ from these quantities. Thus, the system of strategies for Gambler is exactly the set of all functions

$$\sigma \colon \mathbb{R} \times (\mathbb{R} \times \{-1, 1\})^{<\mathbb{N}} \to \mathbb{R}$$

such that for all $n \in \mathbb{N}$ and B we have $-B \leqslant \sigma(B, \langle \vec{b} \restriction n, \vec{p} \restriction n \rangle) \leqslant B$.

When it is Casino's turn to play, there will be:

- the current balance $B_n \in \mathbb{R} \cap [0, \infty)$;
- a sequence of bets by Gambler, $\vec{b} \restriction n+1 = \langle b_0, \dots, b_{n-1}, b_n \rangle \in \mathbb{R}^{n+1}$;
- a sequence of digits played by Casino, $\vec{p} \restriction n = \langle p_0, \dots, p_{n-1} \rangle \in \{-1, 1\}^n$;

and the strategy should produce the next digit $p_n = \pm 1$. Thus, the system of strategies for Casino is exactly the set of all functions

$$\tau \colon \mathbb{R} \times \bigcup_{n \in \mathbb{N}} \bigl(\mathbb{R}^{n+1} \times \{-1, 1\}^n\bigr) \to \{-1, 1\}.$$

In fact, the value of B_n is completely determined by $\vec{b} \restriction n$ and $\vec{p} \restriction n$, so including the balance in the domain of the strategies is not necessary. However, all of the strategies that we employ below depend only on B_n and $\vec{p} \restriction n$, so for this paper, on the contrary, including $\vec{b} \restriction n$ in the domain of the strategies is not necessary.

Note that we have built into Gambler's strategies the rule that Gambler cannot 'cheat' by betting more than the current balance. Still we need the following additional definition related to Casino's strategies. A strategy τ for Casino will be called *admissible*, if for every sequence $\vec{b}$ of legal bets by Gambler, Casino obtains $\vec{p} \in P$ following τ.

We say that Gambler *follows a strategy* σ, if for each turn n Gambler bets $b_n = \sigma\bigl(B_n, \langle\langle b_0, \dots, b_{n-1}\rangle, \langle p_0, \dots, p_{n-1}\rangle\rangle\bigr)$. Likewise, Casino *follows a strategy* τ if for each n Casino plays $p_n = \tau\bigl(B_n, \langle\langle b_0, \dots, b_n\rangle, \langle p_0, \dots, p_{n-1}\rangle\rangle\bigr)$.

Example 1. We take the following countable dense set as P:

$$P_1 = \{\vec{p} \in \mathbb{D} : \exists\, N\ \forall\, n \geqslant N\ (p_n = 1)\}.$$

Then in the game $\Gamma(P_1)$ Gambler can increase his initial balance unboundedly. Namely, in this case the strategy for Gambler that sets $b_n = B_n/2$ will lead to $\lim_{n\to\infty} B_n = \infty$, and hence will win the game $\Gamma(P_1, \infty)$. Indeed, as $B_n > 0$ for all n, and since Casino must produce $\vec{p} \in P_1$ when Gambler follows this strategy, he will be paid on a cofinite set: if N is where $\vec{p}$ becomes constant 1 starting from the Nth position, then

$$\lim_{n\to\infty} B_n = \frac{B_N}{2} + \frac{B_N}{2} + \frac{B_N}{2} + \dots = \infty.$$

We summarize the above example by saying that, in $\Gamma(P_1)$, Gambler has a strategy that guarantees $\lim_{n\to\infty} B_n = \infty$. The precise meaning of this is that Gambler has a certain strategy σ such that for every $\vec{p} \in P_1$, when Gambler follows σ and Casino plays a sequence $\vec{p} \in P_1$, the balances converge to infinity. Below we shall state our results using this terminology. For example, the assertion that Casino has a strategy (in $\Gamma(P)$) that guarantees $\sup_{n\in\mathbb{N}} B_n < 1/\lambda(P)$ means that Casino has a strategy τ such that whenever Gambler makes legal bets (so that $|b_n| \leqslant B_n$) and Casino follows τ, a sequence $\vec{p} \in P$ is produced such that for some $\varepsilon > 0$ we have $B_n \leqslant 1/\lambda(P) - \varepsilon$ for all n.

For games similar to $\Gamma(P, H)$ we say that Gambler *wins* $\Gamma(P, H)$ if Gambler has a winning strategy, that is, a strategy that guarantees $\sup_{n\in\mathbb{N}} B_n > H$ (or $\sup_{n\in\mathbb{N}} B_n = \infty$ when $H = \infty$). Correspondingly, we say that Casino *wins* $\Gamma(P, H)$ if Casino has a strategy that guarantees $\sup_{n\in\mathbb{N}} B_n \leqslant H$ (or $\sup_{n\in\mathbb{N}} B_n < \infty$ when $H = \infty$).

We note here that if Casino has a strategy that guarantees a small limit of the balances in some sense, then this same strategy will also guarantee that the supremum of the balances is equally small, since Gambler is allowed to bet zero on any turn, which preserves the achieved balance. When we make a claim that one of the two players has a strategy guaranteeing some inequality (or equality) involving $\lim_{n\to\infty} B_n$, we mean here that the limit exists and satisfies this inequality.

§ 2. Two key strategies

We now describe two special strategies for Gambler and Casino, which play a crucial role in this game. The main idea is that Gambler would like to be able to bet a certain amount so that, after Casino plays, Gambler is no worse off than he was at the start of the turn. When it is Gambler's turn to play, the position of the game can be described by the current balance $B = B_n$ and the initial segment $s = \vec{p} \restriction n = \langle p_0, \dots, p_{n-1} \rangle$ of the infinite sequence $\vec{p}$ played by Casino. For $s \in \mathbf{S}$, we denote by $\lambda_s(P)$ the relative measure of the set P in $\mathbb{D}_s$:

$$\lambda_s(P) = \frac{\lambda(P \cap \mathbb{D}_s)}{\lambda(\mathbb{D}_s)} = \lambda(P \cap \mathbb{D}_s)\, 2^{\operatorname{lh} s} \in [0,1] \cap \mathbb{R}.$$

It is now reasonable to define the *quality* of the position $\langle B, s \rangle$ to be equal to $B/\lambda_s(P)$ (at least when P is measurable). The reason of such a definition is as follows: in this position Gambler has B dollars to bet on an event of probability $\lambda_s(P)$, and both the increase of B and the decrease of $\lambda_s(P)$ (so that Casino's moves become more predictable) improve Gambler's standing. Hence we make the following definition.

Definition 2. For $s \in \mathbf{S}$, $B \in \mathbb{R}$, and $P \subseteq \mathbb{D}$ the P-*quality* of the position $\langle B, s \rangle$ is defined by

$$Q_s^B(P) = \frac{B}{\lambda_s(P)} = \frac{B}{\lambda(P \cap \mathbb{D}_s) 2^{\operatorname{lh} s}},$$

where λ denotes Lebesgue measure whenever P is measurable and Lebesgue outer measure otherwise.

Note that the quality is always not smaller than the balance. From the position $\langle B, s\rangle$ one can derive the two possible qualities of the next position. If Gambler bets b and Casino plays $p_n = -1$, then the new quality is

$$q_{-1}(b) = \frac{B-b}{\lambda_{s^\frown(-1)}(P)} ,$$

while the response $p_n = 1$ of Casino yields the quality

$$q_1(b) = \frac{B+b}{\lambda_{s^\frown 1}(P)} .$$

In the case where neither of the two relative measures of P is zero, using the fact that $2\lambda_s(P) = \lambda_{s^\frown 1}(P) + \lambda_{s^\frown(-1)}(P)$, we see that there is a unique bet

$$b^* = B\frac{\lambda_{s^\frown 1}(P) - \lambda_{s^\frown(-1)}(P)}{\lambda_{s^\frown 1}(P) + \lambda_{s^\frown(-1)}(P)} ,$$

which will give

$$q_{-1}(b^*) = q_1(b^*) = Q^B_s(P)$$

and hence preserve the quality regardless of Casino's answer. For all the other bets $b \in [-B, B]$, one of the two resulting qualities $q_i(b)$ is strictly less than the quality Q^B_s of the original position (correspondingly, the second resulting quality $q_{-i}(b)$ is greater than Q^B_s). This is illustrated by Fig. 1, where we assume that $\lambda_{s^\frown 1}(P) > \lambda_{s^\frown(-1)}(P)$.

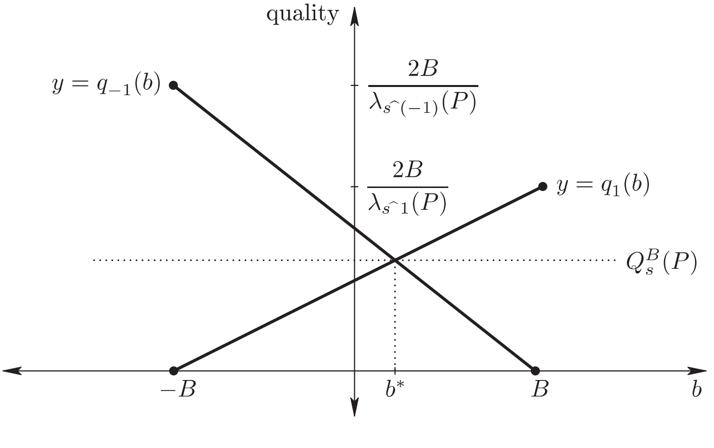

Figure 1

Thus, we see that when both relative measures are positive, it is always possible for Gambler to make an admissible bet $b_n = b^*$ so that the new quality is precisely equal to the quality of the original position. However, when one of the relative measures is zero, Gambler does not have as much control and, as pointed out below, may be forced to accept a decrease in quality. This strategy deserves a special name.

Definition 3. For any measurable set $G \subseteq \mathbb{D}$ the strategy for Gambler in the game $\Gamma(P)$ that is defined by

$$b_n = b_n^* = B_n \frac{\lambda_{(\vec{p}\restriction n)^\frown 1}(G) - \lambda_{(\vec{p}\restriction n)^\frown(-1)}(G)}{\lambda_{(\vec{p}\restriction n)^\frown 1}(G) + \lambda_{(\vec{p}\restriction n)^\frown(-1)}(G)}$$

will be called the *G-quality preserving strategy* and will be denoted by σ_G.

The set G in this definition should be viewed as an approximation to the actual pay-off set P determining the game. The strategies for Gambler that we shall consider will be combinations of G-quality preserving strategies for appropriate approximations G of the set P, but as a rule, the P-quality preserving strategy itself will not be used. Note that σ_G depends only on the sequence $\vec{p} \restriction n$ of previous moves played by Casino and, of course, also on the current balance B_n. Thus, Gambler is free to switch from σ_G to some other $\sigma_{G'}$ at any point of the game, then return to σ_G if necessary at a later point, or switch to $\sigma_{G''}$ for yet another approximation G'', and so on.

These quality preserving strategies have one major flaw: they are not good in positions $\langle B, s\rangle$ such that either $\lambda_{s^\frown 1}(G)$ or $\lambda_{s^\frown(-1)}(G) = 0$. By definition the strategy σ_G bets everything on the opposite event. For instance, suppose that $\lambda_{(\vec{p}\restriction n)^\frown 1}(G) = 0$ on turn n, but $P \cap \mathbb{D}_{(\vec{p}\restriction n)^\frown 1} \neq \varnothing$. Then the strategy σ_G will inform Gambler to play $b_n = -B_n$, that is, to bet everything on the move $p_n = -1$ of Casino. Now Casino may choose $p_n = 1$, collect all Gambler's money on this turn and comfortably finish the game leaving Gambler with zero balance and the impossibility of non-zero bets. (Casino is free to continue the game with the only aim to obtain $\vec{p} \in P \cap \mathbb{D}_{(\vec{p}\restriction n)^\frown 1}$ at the end, which is possible since it is assumed that $P \cap \mathbb{D}_{(\vec{p}\restriction n)^\frown 1} \neq \varnothing$.) Note, however, that when G is an open set containing P, this situation never arises.

The notion of quality also provides a strategy of fundamental importance for Casino. The main idea is as follows. Any strategy τ for Casino should at least preserve the chance to stay within the pay-off set P. Therefore, whenever the players reach a position where $P \cap \mathbb{D}_{(\vec{p}\restriction n)^\frown i} = \varnothing$, the strategy τ must play $p_n = -i$. (By the way, in this position Gambler can bet the whole of the balance on $p_n = -i$.) When both extensions $(\vec{p} \restriction n)^\frown i$, $i = \pm 1$, give positive relative measures for P and Gambler bets the P-quality preserving amount, then Casino can play by a move b_n equal to 1 or -1, with the same effect. When both extensions give positive relative measures for P, but Gambler bets something different from the P-quality preserving amount, then Casino can play so that the P-quality of the position decreases.

Definition 4. For any Lebesgue measurable set $F \subseteq \mathbb{D}$, the strategy for Casino which plays $p_n = 1$ whenever one of the following conditions is satisfied:

1) $\lambda_{(\vec{p}\restriction n)^\frown(-1)}(F) = 0$;
2) b_n is the F-quality preserving bet and $\lambda_{(\vec{p}\restriction n)^\frown 1}(F) \geqslant \lambda_{(\vec{p}\restriction n)^\frown(-1)}(F)$,
3) b_n is not the F-quality preserving bet and

$$\frac{B_n + b_n}{\lambda_{(\vec{p}\restriction n)^\frown 1}(F)} < \frac{B_n - b_n}{\lambda_{(\vec{p}\restriction n)^\frown(-1)}(F)}\,;$$

and plays $p_n = -1$ in all other cases will be called the *F-quality decreasing strategy* for Casino and will be denoted by τ_F.

We can observe that τ_F decreases the F-quality whenever Gambler's previous bet is not F-quality preserving in the above-defined sense. Again, Casino is free to switch from τ_F to another $\tau_{F'}$ at any time, or even change the strategy infinitely often. By the way, when $F \subseteq P$ is a closed set, τ_F is an admissible strategy, which means that it will produce $\vec{p} \in F$, because τ_F always produces a digit p_n with $F \cap \mathbb{D}_{s^\frown p_n} \neq \varnothing$, so $\vec{p} \in F$ since the set F is closed.

To get a feel of how these strategies behave, we give an example in which, quite surprisingly, σ_P is not the best of Gambler's strategies.

Example 5. For $n \in \mathbb{N}$ let $s_n = \langle (-1)^{2n+1}, 1 \rangle$ be the finite sequence starting with $2n+1$ copies of -1 followed by a single copy of 1. Let $-\vec{1}$ be the constant infinite sequence of -1's. The set $P = \langle -\vec{1} \rangle \cup \bigcup_{n\in\mathbb{N}} \mathbb{D}_{s_n}$ is closed and has measure $1/4 + 1/16 + 1/64 + \cdots = 1/3$. Let us show that Casino has an admissible strategy τ in $\Gamma(P)$, which guarantees $\sup_{n\in\mathbb{N}} B_n < 1/\lambda(P) = 3$.

Initially, we have a position of quality $\frac{1}{1/3} = 3$ and we have $\lambda_{\langle 1 \rangle}(P) = 0$ and $\lambda_{\langle -1 \rangle}(P) = 2/3$. It is clear that Casino must play $p_0 = -1$ for otherwise $\vec{p} \notin P$. Therefore, the strategy σ_P calls for

$$b_0^* = B_0 \frac{\lambda_{\langle 1 \rangle}(P) - \lambda_{\langle -1 \rangle}(P)}{\lambda_{\langle 1 \rangle}(P) + \lambda_{\langle -1 \rangle}(P)} = 1 \cdot \frac{0 - 2/3}{0 + 2/3} = -1$$

(that is, the bet of the whole of the initial balance $B_0 = 1$ on the move $p_n = -1$). The required strategy τ for Casino selects $p_0 = -1$, and the new balance becomes $B_1 = 1 - b_0 \leqslant 2$.

If Gambler in fact bets $b_0 = -1$ following σ_P, then the new P-quality remains equal to

$$\frac{B_1}{\lambda_{\langle -1 \rangle}(P)} = \frac{2}{2/3} = 3.$$

After this the two relative measures of P are equal to

$$\lambda_{\langle -1,1 \rangle}(P) = 1, \qquad \lambda_{\langle -1,-1 \rangle}(P) = 4\left(\frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \cdots\right) = \frac{1}{3}.$$

Thus, the P-quality preserving bet of Gambler will be

$$b_1^* = B_1 \frac{1 - 1/3}{1 + 1/3} = \frac{B_1}{2}.$$

Should Gambler bet any amount b_1 satisfying $b_1 < B_1/2$, τ plays $p_1 = 1$, and we obtain $\vec{p} \in P$ independently of the following moves in the game, so that Casino can concentrate on reducing Gambler's balance. After the next pair of moves (with $b_1 < B_1/2$), there will be $B_2 < 3$ and Casino can reduce the balance whenever Gambler bets any non-zero amount in the future.

However, if Gambler bets $b_1 \geqslant B_1/2$ on turn 1, then τ calls for the bet $p_1 = -1$. In this case the new balance will be equal to $B_2 \leqslant 1$, and we are essentially back at the start of the game with balance no greater than the original balance $B_0 = 1$. That is, the balance has not increased, and the pay-off set P on the Baire interval $\mathbb{D}_{\langle -1,-1 \rangle}$ looks identical to the pay-off set at the start of the game. Thus, Casino can continue to play in accordance with the same plan. This outlines the strategy τ.

Should Gambler follow the P-quality preserving strategy on each turn, that is, $b_n = B_n/2$, the balances will be

$$B_n = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 2 & \text{if } n \text{ is odd,} \end{cases}$$

while the above-defined Casino's strategy τ will produce $\vec{p} = -\vec{1} \in P$.

If on any odd turn n Gambler bets $b_n < B_n/2$, then τ plays $p_n = 1$ and, as above, the current balance will be strictly less than 3 and can only decrease whenever Gambler makes a non-zero bet in the future.

Should Gambler ever bet $b_n > B_n/2$ on any odd turn, τ would play $p_n = -1$ and the quality decreases to $3 - \varepsilon$ for some $\varepsilon > 0$. Then Casino can maintain a quality (and hence a balance) of $3 - \varepsilon$ or less for the rest of the game.

Thus, in all cases the strategy τ guarantees that the balances are always strictly less than 3. As far as Gambler is concerned, his optimal strategy is simply to bet $b_0 = -1$ at the start, then $b_1 = 1 - \varepsilon$, and $b_n = 0$ for $n \geqslant 2$. By doing so, Gambler continues with a balance of $3 - \varepsilon$ dollars during the course of the game, which, for small ε, looks much better than the permanent oscillations between 1 and 2 dollars that occur when Casino follows the strategy τ and Gambler follows the P-quality preserving strategy.

§ 3. Case of measure zero

In this section we characterize zero-measure sets as the class of all sets $P \subseteq \mathbb{D}$ such that Gambler has a strategy in $\Gamma(P)$ that guarantees $\sup_{n\to\infty} B_n = \infty$. A similar characterization of zero-measure sets was discovered in [4] (see also [2] and [5] on related results). We use $\lambda^+(P)$ to denote Lebesgue outer measure of a set $P \subseteq \mathbb{D}$, that is, the infimum of measures of the open sets covering P.

Theorem 6. *Let $P \subseteq \mathbb{D}$.*

1. *If $\lambda^+(P) = 0$, then Gambler wins $\Gamma(P, \infty)$, that is, Gambler has a strategy in $\Gamma(P)$ that even guarantees $\lim_{n\to\infty} B_n = \infty$.*
2. *If Gambler wins $\Gamma(P, \infty)$, then $\lambda^+(P) = 0$.*

Proof. 1. When P has measure zero, Casino is forced to produce $\vec{p}$ from a very limited set. Therefore, it is reasonable to assume that Casino's moves will become sufficiently predictable so that Gambler will be able to capitalize on this. The quality preserving strategies convert this assumption into a precise result. Let $\langle G_n \rangle_{n\in\mathbb{N}}$ be a decreasing sequence of open sets such that

(i) $P \subseteq G_n$ for each $n \in \mathbb{N}$;

(ii) $\lim_{n\to\infty} \lambda(G_n) = 0$.

The required Gambler's strategy can be described as follows.

Let m_0 be the least number such that $\lambda(G_{m_0}) \leqslant 1/2$. Gambler begins by keeping half of his initial 1 dollar in a 'bankroll', pretending his balance is only 1/2 dollar. Gambler follows $\sigma_{G_{m_0}}$, that is, the G_{m_0}-quality preserving strategy, until after the first turn n_0 when Casino has produced a finite sequence

$$s_{n_0} = \langle p_0, p_1, \dots, p_{n_0} \rangle \in \{-1, 1\}^{n_0+1}$$

such that $\mathbb{D}_{s_{n_0}} \subseteq G_{m_0}$. This must happen since the entire infinite sequence $\vec{p}$ must belong to the set P, and therefore to the open set G_{m_0} that covers P. Note that by definition the intersection $P \cap \mathbb{D}_{s_{n_0}}$ is non-empty.

Following the strategy $\sigma_{G_{m_0}}$ on this first series of turns Gambler obtains the G_{m_0}-quality of the final position equal to the original G_{m_0}-quality:

$$\frac{B_{n_0}}{\lambda_{s_{n_0}}(G_{m_0})} = \frac{B_{n_0}}{1} = \frac{1/2}{\lambda(G_{m_0})} \geqslant \frac{1/2}{1/2} = 1,$$

therefore the balance B_{n_0} is at least equal to 1 and in fact even to $1 + 1/2 = 3/2$ since Gambler still has $1/2$ dollar in the bank. By the same reason the actual Gambler's balance is at least $1/2$ on each of these first turns.

Now let m_1 be the least integer $> m_0$ such that $\lambda_{s_{n_0}}(G_{m_1}) \leqslant 1/2$. Note that in fact $\lambda_{s_{n_0}}(G_{m_1}) > 0$, as otherwise the open set $\mathbb{D}_{s_{n_0}} \cap G_{m_1}$ would be empty, and the set $\mathbb{D}_{s_{n_0}} \cap P$ would also be empty, contrary to what we said above.

Starting the second series of turns Gambler now reserves at least 1 dollar, keeps precisely $1/2$ dollar for gambling, and follows the G_{m_1}-quality preserving strategy $\sigma_{G_{m_1}}$ until after the first turn $n_1 > n_0$ when the sequence

$$s_{n_1} = \langle p_0, p_1, \ldots, p_{n_1} \rangle \in \{-1, 1\}^{n_1+1}$$

of Casino's plays satisfies $\mathbb{D}_{s_{n_1}} \subseteq G_{m_1}$. Such n_1 exists by the same reasons as above, and the intersection $P \cap \mathbb{D}_{s_{n_1}}$ is non-empty. After turn n_1, since the G_{m_1}-quality does not decrease, Gambler's balance will be at least 1 dollar because

$$\frac{B_{n_1}}{\lambda_{s_{n_1}}(G_{m_1})} = \frac{B_{n_1}}{1} = \frac{1/2}{\lambda_{s_{n_0}}(G_{m_1})} \geqslant \frac{1/2}{1/2} = 1,$$

but in fact it will be at least 2 dollars together with the reserved amount of at least 1 dollar. Note that in the course of this second series of turns (from n_0 to n_1) the actual balance will always be at least 1.

We consider now the least integer $m_2 > m_1$ such that $\lambda_{s_{n_1}}(G_{m_2}) \leqslant 1/2$; then $\lambda_{s_{n_1}}(G_{m_2}) > 0$, as above. In the third series of turns Gambler reserves $\geqslant 3/2$ dollars, still keeping precisely $1/2$ dollar for gambling, and follows the G_{m_2}-quality preserving strategy until after the first turn $n_2 > n_1$ when $\mathbb{D}_{s_{n_2}} \subseteq G_{m_2}$. Then the balance will increase by at least $1/2$ dollar; and so on.

It is clear that following such a strategy Gambler ensures the inequality

$$\lim_{n\to\infty} B_n = \infty.$$

2. Suppose that Gambler has a strategy σ guaranteeing that independently of Casino playing any sequence $\vec{p} \in P$ in $\Gamma(P)$ the balances grow to ∞, that is, $\sup_{n\in\mathbb{N}} B_n = \infty$. For any finite sequence $s \in \mathbf{S}$ let $C_\sigma(s)$ be the balance that results from Gambler following the strategy σ and Casino producing the sequence of plays s (after the number of turns equal to the length of s). Clearly, the quantities $C_\sigma(s^\frown 1)$ and $C_\sigma(s^\frown(-1))$ are equal to $C_\sigma(s) + b$ and $C_\sigma(s) - b$, respectively, where $b = \sigma(C_\sigma(s), \langle \vec{b}_s, s \rangle)$ (the bet that σ tells Gambler to make), and $\vec{b}_s$ is the sequence of previous bets made by Gambler. Therefore,

$$C_\sigma(s^\frown 1) + C_\sigma(s^\frown(-1)) = 2C_\sigma(s),$$

and this immediately implies that $\sum_{s\in\{-1,1\}^n} C_\sigma(s) = 2^n$.

Now let $S_n = \{s \in \mathbf{S} : C_\sigma(s) > n\}$. Obviously, the set $G_n = \bigcup_{s \in S_n} \mathbb{D}_s$ is open and satisfies $P \subset \bigcap_{n\in\mathbb{N}} G_n$, and if we can show that $\lambda(G_n) \leqslant 1/n$, then P will have outer measure 0. Assume by contradiction that $\lambda(G_n) > 1/n$. Each open set $G \subseteq \mathbb{D}$ can be approximated by its clopen subsets of measure arbitrarily close to $\lambda(G)$. Thus, we can find a finite system of sets $S' \subseteq S_n$ such that:

- if $s \neq t$ belong to S', then $\mathbb{D}_s \cap \mathbb{D}_t = \varnothing$;
- $\lambda\bigl(\bigcup_{s\in S'} \mathbb{D}_s\bigr) = \sum_{s\in S'} 2^{-\operatorname{lh} s} > 1/n$.

If $s \in \{-1,1\}^n$, then the balances $C_\sigma(t)$ of the extensions t of length $n+k$ of the sequence s in $\mathbf{S}$ obviously add up to $2^k C_\sigma(s)$. We put $\ell = \max_{s\in S'}\{\operatorname{lh} s\}$ and consider the set

$$T = \bigl\{t \in \{-1,1\}^\ell : \exists\, s \in S' \ (s \subseteq t)\bigr\} \subseteq \{-1,1\}^\ell$$

of all finite sequences of length ℓ that are extensions of some element in S'. Then

$$\begin{aligned} 2^\ell &= \sum_{s\in\{-1,1\}^\ell} C_\sigma(s) \geqslant \sum_{t\in T} C_\sigma(t) = \sum_{s\in S'} 2^{\ell-\operatorname{lh} s} C_\sigma(s) \\ &= 2^\ell \sum_{s\in S'} 2^{-\operatorname{lh} s} C_\sigma(s) > 2^\ell \frac{1}{n} n = 2^\ell, \end{aligned}$$

a contradiction. Thus, each set G_n has measure at most $1/n$, and therefore P has outer measure 0.

Corollary 7. *For $P \subseteq \mathbb{D}$ the following conditions are equivalent*:

- $\lambda(P) = 0$;
- *Gambler wins $\Gamma(P,\infty)$, that is, has a strategy in the game $\Gamma(P)$ that guarantees $\lim_{n\in\mathbb{N}} B_n = \infty$.*

§ 4. The case of positive measure from Gambler's point of view

In this section we verify that Gambler can always increase his balance to about $1/\lambda^+(P)$, and conversely, that optimal earnings by Gambler produce upper bounds for the outer measure of the (not necessarily measurable) set P. In principle, this is consistent with Theorem 6 if we set $1/0 = \infty$.

Theorem 8. *Let $P \subseteq \mathbb{D}$ and suppose that $0 < h \leqslant 1$.*

1. *If $\lambda^+(P) \leqslant h$ then for any $\varepsilon > 0$ Gambler wins $\Gamma(P, h^{-1} - \varepsilon)$ and even has a strategy in $\Gamma(P)$ that guarantees $\lim_{n\to\infty} B_n > h^{-1} - \varepsilon$. In particular, if $\lambda^+(P) < h$, then Gambler wins $\Gamma(P, h^{-1})$.*
2. *If Gambler wins $\Gamma(P, h^{-1})$, then $\lambda^+(P) < h$. In particular, if Gambler wins $\Gamma(P, h^{-1} - \varepsilon)$ for each $\varepsilon > 0$, then $\lambda^+(P) \leqslant h$.*

Proof. 1. Here we must prove both the existence of the limit and the fact that it is greater than $h^{-1} - \varepsilon$. The idea of the proof is roughly the same as in the first part of the proof of Theorem 6. If P is open, then the P-quality preserving strategy yields

$$\frac{B_n}{\lambda_{\vec{p}\restriction n}(P)} = \frac{1}{\lambda(P)} \geqslant h^{-1},$$

and the relative measure in the denominator ($\lambda_{\vec{p}\restriction n}(P)$) will eventually become 1. The small number ε is necessary to reduce the general case to the case of open sets since quality preserving strategies produce reliably the expected result only in the case of open sets.

Fix $\varepsilon > 0$. If either $h = 1$, or $h < 1$ and $h^{-1} - \varepsilon \leqslant 1$ simultaneously, then Gambler can bet zero on each turn of the game. Therefore, we may assume that $h < 1$ and $h^{-1} - \varepsilon > 1$, so that $h < h/(1 - \varepsilon h) < 1$. In this case there is an open set G such that $P \subseteq G$ and $\lambda(G) < h/(1 - \varepsilon h)$. Assume that Gambler plays in accordance with σ_G. Eventually, Casino's plays p_n will produce a finite sequence s_{n_0} such that $\mathbb{D}_{s_{n_0}} \subseteq G$. Since the G-quality does not change, we shall have

$$\frac{B_0}{\lambda(G)} = \frac{1}{\lambda(G)} = \frac{B_{n_0}}{\lambda_{s_{n_0}}(G)} = \frac{B_{n_0}}{1}\,.$$

Thus, $B_{n_0} = 1/\lambda(G) > (1-\varepsilon h)/h = h^{-1} - \varepsilon$. For all the following turns from $n_0 + 1$ onwards Gambler can simply bet zero, and then $\lim_{k\to\infty} B_k = B_{n_0} > h^{-1} - \varepsilon$, as required.

2. The second statement of this part of the theorem is an easy consequence of the first, so we shall only verify the first statement. The idea is as in the proof of Theorem 6: any Gambler's strategy can make individual balances large, but when one sums all the balances after n turns of the game corresponding to $s \in \{-1, 1\}^n$, then the total is exactly 2^n, and there cannot be many finite sequences s of the form $\vec{p} \restriction n$ for $\vec{p} \in P$.

Let σ be a strategy for Gambler that guarantees that at least $\sup_{n\in\mathbb{N}} B_n > h^{-1}$. For $s \in \mathbf{S}$ let $C_\sigma(s)$ denote the balance that results from Gambler following the strategy σ and Casino playing a (finite) sequence of plays s. Let

$$S_h = \{s \in \mathbf{S} : C_\sigma(s) > h^{-1}\},$$

so that $P \subseteq G = \bigcup_{s\in S_h} \mathbb{D}_s$, and it is sufficient to show that $\lambda(G) < h$. Assume the converse, so that $\lambda(G) \geqslant h$. Consider an arbitrary sequence $s_0 \in S_h$. Then for some $\varepsilon_0 > 0$ we have $C_\sigma(s_0) = h^{-1} + \varepsilon_0$. It is clear that if h' is close to h and $B \geqslant h^{-1}$, then $h' \cdot B$ is close to 1. We take $\delta > 0$ such that if $h' \geqslant h - \delta$, then $h' \cdot h^{-1} > 1 - \varepsilon_0 2^{-\operatorname{lh} s_0}$. We can find a finite subsystem $S' \subseteq S_h$ such that:

- $s_0 \in S'$;
- if $s \neq t \in S'$, then $\mathbb{D}_s \cap \mathbb{D}_t = \varnothing$;
- $\sum_{s\in S'} \lambda(\mathbb{D}_s) = \sum_{s\in S'} 2^{-\operatorname{lh} s} > h - \delta$.

Let $\ell = \max_{s\in S'}\{\operatorname{lh} s\}$; then

$$\begin{aligned} 2^\ell = \sum_{s\in 2^\ell} C_\sigma(s) &\geqslant \sum_{s\in S'} 2^{\ell-\operatorname{lh} s} C_\sigma(s) = 2^\ell \sum_{s\in S'} 2^{-\operatorname{lh} s} C_\sigma(s) \\ &> 2^\ell\big[(h-\delta)h^{-1} + \varepsilon_0 2^{-\operatorname{lh} s_0}\big] > 2^\ell, \end{aligned}$$

a contradiction. Thus, $\lambda(G) < h$, and therefore $\lambda^+(P) < h$ since $P \subseteq G$.

Corollary 9. *For $P \subseteq \mathbb{D}$ and $0 < h \leqslant 1$ the following conditions are equivalent*:

- $\lambda^+(P) \leqslant h$;
- *for each $\varepsilon > 0$ Gambler wins $\Gamma(P, h^{-1} - \varepsilon)$, that is, has a strategy in the game $\Gamma(P)$ that guarantees* $\sup_{n\in\mathbb{N}} B_n > h^{-1} - \varepsilon$.

§ 5. The case of positive measure from Casino's point of view

In this section we characterize the inner measure $\lambda^{-}(P)$ of a set P in terms of optimal strategies for Casino. We show that Casino can always keep the supremum of Gambler's balances below or equal to the reciprocal of the inner measure of P, and optimal strategies for Casino produce lower bounds for the inner measure of P.

Theorem 10. *Let $P \subseteq \mathbb{D}$ and $h \in (0,1]$.*

1. *If $\lambda^{-}(P) \geqslant h$, then Casino wins the game $\Gamma(P, h^{-1})$, that is, it has an admissible (ensuring $\vec{p} \in P$) strategy in $\Gamma(P)$ that guarantees $\sup_{n\in\mathbb{N}} B_n \leqslant h^{-1}$.*
2. *If Casino wins the game $\Gamma(P, h^{-1}+\varepsilon)$ for each $\varepsilon > 0$, then $\lambda^{-}(P) \geqslant h$.*

Proof. 1. The idea is to employ the F-quality decreasing strategies for Casino for appropriate large closed sets $F \subseteq P$. To begin with, note that $\lambda^{-}(P) \geqslant h$, and therefore there is a set $U = \bigcup_{n\in\mathbb{N}} F_n \subseteq P$, where $\langle F_n \rangle_{n\in\mathbb{N}}$ is an increasing sequence of closed sets with $\lambda(F_0) > 0$ and $h = \lambda(U) = \lim_{n\to\infty} \lambda(F_n)$. The required strategy for Casino can be defined as follows.

The whole course of the game is split into two parts. The first part is that initial segment of the game (which can be empty, or, the other way around, can contain the entire game) on which Gambler follows the U-quality preserving strategy σ_U. Casino's moves do not affect the U-quality of positions, which remains the same during this part of the game, hence Casino can play while taking into account arguments related to the closed set F_0 rather than the set U. Namely, during this part Casino will respond by plays $p_n = \pm 1$ defined so that $\lambda_{s^\frown p_n}(F_0) \geqslant \lambda_s(F_0)$. To be more precise, if Gambler plays

$$b_n = B_n \frac{\lambda_{(\vec{p}\restriction n)^\frown 1}(U) - \lambda_{(\vec{p}\restriction n)^\frown(-1)}(U)}{\lambda_{(\vec{p}\restriction n)^\frown 1}(U) + \lambda_{(\vec{p}\restriction n)^\frown(-1)}(U)}$$

(the U-quality preserving bet), then Casino answers by

$$p_n = \begin{cases} 1, & \text{if } \lambda_{(\vec{p}\restriction n)^\frown 1}(F_0) \geqslant \lambda_{(\vec{p}\restriction n)^\frown(-1)}(F_0), \\ -1 & \text{otherwise.} \end{cases}$$

If Gambler follows the strategy σ_U for the entire game, then Casino produces a sequence $\vec{p} \in F_0 \subseteq P$ (because F_0 is closed) and, at the same time, the U-quality never changes. Thus, since the balance never exceeds the quality, it follows that $B_n \leqslant h^{-1}$ for all $n \in \mathbb{N}$.

But if Gambler leaves the strategy σ_U at some point, then let n be the first turn where Gambler does not follow σ_U. Here the second part of the game starts, and now Casino changes its strategy, too. Namely, Casino can choose its next move $p_n = \pm 1$ so that the new U-quality will be strictly less than the original one. Then there is $\varepsilon > 0$ such that

$$\frac{B_{n+1}}{\lambda_{\vec{p}\restriction(n+1)}(U)} = h^{-1} - 2\varepsilon.$$

And since $\lim_{n\to\infty} \lambda(F_n) = \lambda(U)$, Casino can choose $k \in \mathbb{N}$ so that

$$\frac{B_{n+1}}{\lambda_{\vec{p}\restriction(n+1)}(F_k)} < h^{-1} - \varepsilon.$$

Suppose Casino follows the F_k-quality decreasing strategy from this point on. In this case the F_k-quality of positions will not increase, therefore the balances, bounded above by the quality, will satisfy $\sup_{m>n} B_m \leqslant h^{-1} - \varepsilon$. As regards the balances B_0 through B_n (the first part of the game), they are bounded by the original U-quality, h^{-1}. Therefore, following this game plan, Casino always has $\sup_{n\in\mathbb{N}} B_n \leqslant h^{-1}$, while on the other hand we have either $\vec{p} \in F_0$ (when the second part of the game does not exist), or $\vec{p} \in F_k$ (if the second part exists), therefore $\vec{p} \in U$ in both cases, as required.

2. To begin with, we define an analogue $C_\tau(\beta)$ of the balance function $C_\sigma(s)$ introduced above. If τ is an admissible strategy for Casino (that is, τ always produces sequences $\vec{p} \in P$) and $\beta = \langle b_0, \dots, b_{n_1}\rangle$ is the sequence of n initial bets of Gambler, then $C_\tau(\beta)$ denotes the balance after the n turns when Gambler bets $b_0, \dots, b_{n_1}$ and Casino follows the strategy τ.

Lemma 11. *Let $P \subseteq \mathbb{D}$, let τ be an admissible strategy for Casino in $\Gamma(P)$ and $\varepsilon > 0$. Then there is a closed set $P' \subseteq P$, and for each $\vec{p'} \in P'$ there is an infinite sequence of bets $\vec{b}_{\vec{p'}} \in \{-1,1\}^{\mathbb{N}}$ by Gambler such that for each $n \in \mathbb{N}$:*

(i) *if $\vec{p'} \restriction n = \vec{p''} \restriction n$, then $\vec{b}_{\vec{p'}} \restriction n = \vec{b}_{\vec{p''}} \restriction n$;*

(ii) *τ produces $\vec{p'} \restriction n$ when Gambler bets $\vec{b}_{\vec{p'}} \restriction n$;*

(iii) *$\sum_{s\in P'_n} \widetilde{B}_\tau(\vec{b}_s) > 2^n - 2^n\varepsilon$.*

Proof. We shall define P' as the set of branches of the tree $T = \bigcup_{n\in\mathbb{N}} P'_n \subseteq \{-1,1\}^{<\mathbb{N}}$, where $P'_n \subseteq \{-1,1\}^n$. The sequence $\vec{b}_{\vec{p'}}$ of Gambler's bets that corresponds to the sequence $\vec{p'} \in P'$ will be defined by induction, so the sequence of bets corresponding to the sequence $\vec{p'} \restriction n$ will be an initial segment of the sequence corresponding to $\vec{p'} \restriction n+k$, and τ will always be used to produce the responses p'_{n+k}. Hence branches of the tree will correspond to runs of $\Gamma(P)$, where Casino follows τ and Gambler bets a sequence $\vec{b}_{\vec{p'}}$, and therefore $P' \subseteq P$. We shall only be required to determine P'_n (the nodes of the tree of the nth level) from P'_{n-1}. To clarify notation we suppress the dependence of the values of τ on everything but Gambler's current bet (for instance, b_n) since all the other inputs (for instance, previous bets by Gambler) will be clear from the context.

We begin with the definition of the 0th level $P'_0 = \langle\Lambda\rangle$, where Λ is the empty sequence (of length 0). This corresponds to the beginning of the game, and since Gambler makes the initial bet, we can define $\vec{b}_\Lambda = \Lambda$, which leads to the balance equal to the initial balance $1 > 2^0 - \varepsilon$.

Assume now that we have defined the levels $P'_n \subseteq \{-1,1\}^n$ of the required tree for $n = 0, 1, \dots, k-1$ such that:

- for all $n < n' < k$ and $s \in P'_n$ there is at least one sequence $t \in P'_{n'}$ such that $s \subset t$;
- to each $s \in P'_n$ we have associated a sequence of bets $\vec{b}_s$ by Gambler such that
  - when Gambler bets the sequence $\vec{b}_s$, then the strategy τ produces the sequence s of Casino's answers,
  - if the sequences s and t, $s \subset t$, belong to $\bigcup_{n<k} P'_n$, then $\vec{b}_s \subset \vec{b}_t$;
- if $n < k$, then $\sum_{s\in P'_n} C_\tau(\vec{b}_s) > 2^n - 2^n\varepsilon$.

We must now define P'_k so that the properties above survive. For arbitrary fixed $\delta > 0$ we shall show that each sequence $s \in P'_{k-1}$ can be extended to one or two elements in $\{-1, 1\}^k$ that come from concrete bets by Gambler and by Casino following τ, so that the total balance corresponding to either of the extensions is at least $2C_\tau(\vec{b}_s) - \delta$.

Case 1: $\tau(b_k) = 1$ for all bets $b_k \in [-C_\tau(\vec{b}_s), C_\tau(\vec{b}_s)]$, that is, τ recommends the answer 1 regardless of the bet b_k by Gambler. In this case we put in the set of possible extensions T_k only one extension $s^\frown 1$ and set $\vec{b}_{s^\frown 1} = \vec{b}_s{}^\frown C_\tau(\vec{b}_s)$. In other words, Gambler bets all of his money on 1, and since the strategy τ also plays 1, the balance doubles.

Case 2: $\tau(b_k) = -1$ for all bets $b_k \in [-C_\tau(\vec{b}_s), C_\tau(\vec{b}_s)]$. Then we put only one extension $s^\frown(-1)$ in T_k and set $\vec{b}_{s^\frown(-1)} = \vec{b}_s{}^\frown - C_\tau(\vec{b}_s)$. Again, the balance doubles for the same reasons.

Case 3: the strategy τ as a function is not constant on the interval

$$[-C_\tau(\vec{b}_s), C_\tau(\vec{b}_s)].$$

Then there exist two values on this interval, say, $b_k^{(1)}$ and $b_k^{(-1)}$, such that $\tau(b_k^{(i)}) = i$ ($i = \pm 1$) and $|b_k^{(1)} - b_k^{(-1)}| < \delta$.[1] We put both extensions in T_k and set $\vec{b}_{s^\frown i} = \vec{b}_s{}^\frown b_k^{(i)}$ ($i = \pm 1$). The two corresponding changes of the balance sum up to

$$2C_\tau(\vec{b}_s) + b_k^{(1)} - b_k^{(-1)} > 2C_\tau(\vec{b}_s) - \delta.$$

Thus, to define T_k we fix a sufficiently small $\delta > 0$ (see below on the choice of δ) and then add to P'_k one or two extensions of each sequence $s \in P'_{k-1}$, as indicated above, defined for this δ. By the induction hypothesis

$$\sum_{s \in P'_{k-1}} C_\tau(\vec{b}_s) > 2^{k-1} - 2^{k-1}\varepsilon.$$

On the other hand,

$$\sum_{s \in P'_k} C_\tau(\vec{b}_s) > 2 \sum_{s \in P'_{k-1}} C_\tau(\vec{b}_s) - 2^{k-1}\delta$$

by definition. Therefore, there is a sufficiently small real number $\delta > 0$ such that $\sum_{s \in P'_k} C_\tau(\vec{b}_s) > 2^k - 2^k\varepsilon$, as required.

The induction step of the construction obviously preserves all the required properties. This finishes the proof of Lemma 11.

We now return to the proof of claim 2 of Theorem 10. To prove that $\lambda^-(P) \geqslant h$ consider an arbitrary $\delta > 0$. We shall find a closed set $P' \subseteq P$ with $\lambda(P') \geqslant h - \delta$. We may assume that $\delta < h$.

It is clear that to obtain the required set it is sufficient to ensure additionally the inequality $\operatorname{card}(P'_n) \geqslant 2^n(h - \delta)$ for all $n \in \mathbb{N}$ in the construction from the proof of the lemma, where $\operatorname{card} X$ is the cardinality of the finite set X.

[1] Note that these values can be chosen among rational numbers. This will be of some importance below for our study of the rational form of the game.

First, we pick ε_1 such that $0 < \varepsilon_1 < \delta/(h(h-\delta))$, and then we have $\varepsilon_2 = \delta(h^{-1}+\varepsilon_1) - h\varepsilon_1 > 0$. Let τ be an admissible strategy for Casino that guarantees that

$$\sup_{n\in\mathbb{N}} B_n \leqslant h^{-1} + \varepsilon_1.$$

Using Lemma 11 for this strategy τ we find a closed set $P' \subseteq P$ satisfying

$$\sum_{s\in P'_n} C_\tau(\vec{b}_s) > 2^n - 2^n\varepsilon_2$$

for all n. Then

$$\sum_{s\in P'_n} C_\tau(\vec{b}_s) \leqslant (h^{-1}+\varepsilon_1)\,\mathrm{card}\, P'_n\,,$$

and hence

$$\begin{aligned}\mathrm{card}\, P'_n &\geqslant \frac{\sum_{s\in P'_n} C_\tau(\vec{b}_s)}{h^{-1}+\varepsilon_1} > \frac{2^n - 2^n\varepsilon_2}{h^{-1}+\varepsilon_1} = \frac{2^n - 2^n\delta(h^{-1}+\varepsilon_1) + 2^n h\varepsilon_1}{h^{-1}+\varepsilon_1}\\ &= \frac{2^n(h-\delta)(h^{-1}+\varepsilon_1)}{h^{-1}+\varepsilon_1} = 2^n(h-\delta).\end{aligned}$$

This implies that $\lambda^-(P') \geqslant h - \delta$, so that we have proved claim 2 of Theorem 10.

Corollary 12. *If $P \subseteq \mathbb{D}$ and $0 < h \leqslant 1$, then the following conditions are equivalent*:

- $\lambda^-(P) \geqslant h$;
- *Casino wins $\Gamma(P, h^{-1})$, that is, has a strategy in $\Gamma(P)$ that guarantees $\sup_{n\in\mathbb{N}} B_n \leqslant h^{-1}$.*

It is an interesting problem whether $\sup_{n\in\mathbb{N}}$ could be replaced by $\lim_{n\to\infty}$ (that is, by the requirement that the limit exists) in part 1 of Theorem 10. Lebesgue's density theorem [6] can be used to show that our strategy actually ensures the existence of $\lim_{n\to\infty} B_n$ for almost every (in the sense of measure) $\vec{p} \in P$. Indeed, we have $\lim_{n\to\infty} \lambda_{\vec{p}\restriction n}(P) = 1$ for almost all $\vec{p} \in P$, so that in the limit the balance and the quality are equal. Since the qualities form a bounded non-increasing sequence of real numbers, they converge. In spite of this, we were unable to eliminate a zero-measure set and to replace fully $\sup_{n\in\mathbb{N}}$ by $\lim_{n\to\infty}$. We point out the following. If $F_0 \subseteq P$ is a closed set of positive measure such that each point $\vec{p} \in F_0$ is a *density point* of P, that is, $\lim_{n\to\infty} \lambda_{\vec{p}\restriction n}(P) = 1$, then a similar argument shows that either $\lim_{n\to\infty} B_n = h^{-1}$, or $\sup_{n\in\mathbb{N}} B_n < h^{-1}$. Hence, if Gambler can actually force the sequence of balances not to converge (to any limit), then he will forfeit some part of his potential earnings.

Another potential improvement of part 1 of Theorem 8 that one might hope for is eliminating ε, that is, proving that if $\lambda^+(P) \leqslant h$, then Gambler has a strategy in $\Gamma(P)$ guaranteeing $\lim_{n\to\infty} B_n \geqslant h^{-1}$. However, Example 5 gives us a closed set of measure $1/3$ such that Casino guarantees that $\sup_{n\in\mathbb{N}} B_n < 3 = 1/\lambda(P)$. Thus, in fact, ε is necessary. The next result (to save space, we skip the simple proof of it, which follows the constructions of Example 5) shows that a similar set can be defined to have any fixed measure $h \in (0, 1)$.

Proposition 13. *For each real number $h \in (0,1)$ there is a closed set $P_h \subseteq \mathbb{D}$ with $\lambda(P_h) = h$ such that Casino has an admissible strategy in the game $\Gamma(P_h)$ that guarantees* $\sup_{n\in\mathbb{N}} B_n < h^{-1}$.

In the light of this result one might conjecture that part 1 of Theorem 10 can be strengthened as follows:

$$\begin{aligned}&\text{if } \lambda^-(P) \geqslant h, \text{ then Casino has a strategy in } \Gamma(P)\\ &\text{that guarantees } \sup_{n\in\mathbb{N}} B_n < h^{-1}\end{aligned}$$

(the last inequality is now strict). However, if G is an open set with $\lambda(G) = h > 0$, then the G-quality preserving strategy for Gambler guarantees that

$$\lim_{n\to\infty} B_n = h^{-1},$$

so Casino cannot have a strategy ensuring $\sup_{n\in\mathbb{N}} B_n < h^{-1}$. However, it should be noted that part 2 of Theorem 10 is actually stronger with ε than without it.

We finish this section by deducing the Lebesgue measurability of a fixed set P from the existence of certain strategies for Casino in games of the form $\Gamma(P,h)$ for appropriate values of h.

Corollary 14. *Let $P \subseteq \mathbb{D}$ be a set satisfying $\lambda^+(P) > 0$ and let*

$$h = \frac{\lambda^+(P) + \lambda^-(P)}{2}.$$

Then Gambler does not have a winning strategy in the game $\Gamma(P, h^{-1})$. If Casino has a winning strategy in the game $\Gamma(P, h^{-1})$, then the set P is measurable.

Proof. Assuming that Gambler still has a winning strategy in the game $\Gamma(P, h^{-1})$, Theorem 8 implies

$$\lambda^+(P) < \frac{\lambda^+(P) + \lambda^-(P)}{2}, \quad \text{and hence} \quad \frac{\lambda^+(P)}{2} < \frac{\lambda^-(P)}{2},$$

which is a contradiction. Suppose that Casino has a winning strategy. Then Theorem 10 implies that

$$\lambda^-(P) \geqslant \frac{\lambda^+(P) + \lambda^-(P)}{2}, \quad \text{and hence} \quad \frac{\lambda^-(P)}{2} \geqslant \frac{\lambda^+(P)}{2}.$$

This shows that $\lambda^-(P) = \lambda^+(P)$, so P is measurable.

§ 6. A 'rational' modification of the game

Here we study a modification of the game $\Gamma(P)$ that involves restricting Gambler's bets to rational amounts. The modified game turns out to be very similar to $\Gamma(P)$ and allows us to give another proof (assuming the axiom of determinacy) that all subsets of $\mathbb{D}$ are Lebesgue measurable.

Thus, if $P \subseteq \mathbb{D}$ and $0 \leqslant H \leqslant \infty$ (H is not necessarily rational), then we define the games $\Gamma^{\mathbb{Q}}(P)$ and $\Gamma^{\mathbb{Q}}(P,H)$ to be identical to $\Gamma(P)$ and $\Gamma(P,H)$, respectively, except that Gambler must bet rational amounts. To be more precise, the game $\Gamma^{\mathbb{Q}}(P,H)$ still involves two players, identified as Gambler and Casino. The game consists of infinitely many steps (turns) numbered by natural integers n. Gambler starts with a balance of $B_0 = 1$ dollar and on each turn chooses a rational number b_n of absolute value not exceeding his current balance B_n. Then Casino plays $p_n = \pm 1$ and the new value of Gambler's balance becomes

$$B_{n+1} = B_n + b_n \cdot p_n \in \mathbb{Q}.$$

As above, we require Casino to produce a sequence of plays $\vec{p} = \langle p_n \rangle_{n\in\mathbb{N}}$ in P and say that Gambler wins a run of $\Gamma^{\mathbb{Q}}(P,H)$ if $\sup_{n\in\mathbb{N}} B_n > H$. When $H = \infty$, we say that Gambler wins a run in the game $\Gamma^{\mathbb{Q}}(P,\infty)$ if $\sup_{n\in\mathbb{N}} B_n = \infty$. Accordingly, we say that Casino wins a run in $\Gamma^{\mathbb{Q}}(P,H)$ if $\sup_{n\in\mathbb{N}} B_n < H$ (or $\sup_{n\in\mathbb{N}} B_n < \infty$ in the case when $H = \infty$).

Again, we say that Gambler (or Casino) wins the game $\Gamma^{\mathbb{Q}}(P,H)$ if Gambler (Casino, respectively) has a winning strategy in the game, that is, an (admissible) strategy guaranteeing the required inequality between H and $\sup_n B_n$ in $\Gamma^{\mathbb{Q}}(P)$. The notion of admissible strategy for Gambler now includes the additional requirement that all bets are rational numbers.

This modification does not change the rules for Casino. That is, any admissible strategy for Casino in $\Gamma(P)$ is also an admissible strategy for Casino in $\Gamma^{\mathbb{Q}}(P)$. Hence the following result is an easy consequence of Theorem 10.

Theorem 15. *If $P \subseteq \mathbb{D}$ and $h \in (0,1]$, then $\lambda^-(P) \geqslant h$ if and only if Casino wins $\Gamma^{\mathbb{Q}}(P,h^{-1})$.*

Proof. The implication $\Longrightarrow$ is an immediate consequence of Theorem 10. To prove the implication $\Longleftarrow$ assume that τ is a winning strategy for Casino in the game $\Gamma^{\mathbb{Q}}(P,h^{-1})$. If we could conclude that τ is still a winning strategy in $\Gamma(P,h^{-1})$, where Gambler has much more freedom in betting, then Theorem 10 would immediately give us the desired inequality. Unfortunately, we cannot prove this property of strategies. On the other hand, one can easily accommodate our proof of Theorem 10 to the 'rational' modification of the game. (See the footnote on page 1611.)

One could conjecture that the restriction of Gambler's bets to rational numbers gives Casino a certain additional advantage and therefore requires weakening the conclusions of part 1 of Theorem 6 and part 1 of Theorem 8. However, each of these results involves ε (or ∞), and hence in fact a minor modification of the quality preserving strategies allows us to prove the same assertions for the modified game.

Consider an arbitrary open set $P \subseteq \mathbb{D}$. Recall that for $s \in \mathbf{S}$ and $B \in \mathbb{R}$ the bet

$$\sigma_P(B,s) = B\frac{\lambda_{s^\frown 1}(P) - \lambda_{s^\frown(-1)}(P)}{\lambda_{s^\frown 1}(P) + \lambda_{s^\frown(-1)}(P)} \qquad (*)$$

is what the P-quality preserving strategy σ_P calls for; here B is Gambler's balance before this turn and s is the sequence of previous plays by the Casino. As noted before, following this strategy in $\Gamma(P)$ Gambler keeps the P-quality of positions equal to the initial quality $1/\lambda(P)$ during the entire game regardless of Casino's moves. In the 'rational' game $\Gamma^{\mathbb{Q}}(P)$ the strategy σ_P is not necessarily admissible: the values of bets computed in accordance with ($*$) are not necessarily rational. Yet it is clear that, for any small fixed $\delta > 0$, the strategy σ_P can be transformed into a 'rational' (giving only rational values of bets) strategy σ_P^δ such that for any n and any sequence $s \in \{-1,1\}^{n+1}$ of plays by Casino, if Gambler follows σ_P^δ, then the P-quality of the position after $n+1$ turns differs from the value $1/\lambda(P)$ by at most $\delta(1-2^{-n-1})$. Such a strategy σ_P^δ can be called a P-quality δ-preserving strategy, where δ-preservation means preservation up to δ.

The proofs of the next two theorems are similar: one simply uses quality δ-preserving strategies instead of the quality preserving strategies in the proofs of Theorems 6 and 8. We shall give only a sketch of the proof of the second result.

Theorem 16. *For a non-empty set $P \subseteq \mathbb{D}$, $\lambda^+(P) = 0$ if and only if Gambler wins $\Gamma^{\mathbb{Q}}(P,\infty)$.*

Theorem 17. *For $P \subseteq \mathbb{D}$ and $h \in (0,1]$, $\lambda^+(P) \leqslant h$ if and only if Gambler wins $\Gamma^{\mathbb{Q}}(P, h^{-1}-\varepsilon)$ for every $\varepsilon > 0$.*

Proof. To prove the implication $\Longrightarrow$ fix $\varepsilon > 0$. If either $h = 1$, or $h < 1$ and $h^{-1} - \varepsilon/2 \leqslant 1$, then Gambler wins by betting zero on each turn. Therefore, we can assume that $h < 1$ and $h^{-1} - \varepsilon/2 > 1$, so that $h < \frac{h}{1-\varepsilon h/2} < 1$. Then we can find an open set G such that $P \subseteq G$ and the measure $\lambda(G)$ is rational and satisfies $\lambda(G) < \frac{h}{1-\varepsilon h/2}$. If Gambler plays in accordance with any (fixed) G-quality $\varepsilon/2$-preserving strategy, then Casino must eventually produce a sequence s_{n_0} satisfying $\mathbb{D}_{s_{n_0}} \subseteq G$. In this case the G-quality will change by at most $\varepsilon/2$. Thus,

$$\frac{1}{\lambda(G)} - \frac{\varepsilon}{2} < \frac{B_{n_0}}{\lambda_{s_{n_0}}(G)} = \frac{B_{n_0}}{1}\,.$$

Hence

$$B_{n_0} > \frac{1}{\lambda(G)} - \frac{\varepsilon}{2} > \frac{1-\varepsilon h/2}{h} - \frac{\varepsilon}{2} = h^{-1} - \varepsilon.$$

For turns n_0+1 and onwards Gambler can bet zero to obtain the required equality $\lim_{k\to\infty} B_k = B_{n_0} > h^{-1} - \varepsilon$.

To prove the implication $\Longleftarrow$, for any $\varepsilon > 0$ let σ_ε be a winning strategy for Gambler in $\Gamma^{\mathbb{Q}}(P, h^{-1}-\varepsilon)$. Then σ_ε guarantees $\sup_{n\in\mathbb{N}} B_n > h^{-1}-\varepsilon$ also in the game $\Gamma(P)$. By Theorem 8,(2), we obtain $\lambda^+(P) \leqslant h$.

The following analogue of Corollary 14 is used to prove, from the axiom of determinacy, that all subsets of $\mathbb{D}$ are Lebesgue measurable. The proof is based on the following fact: for games of the form $\Gamma^{\mathbb{Q}}(P)$ each player has only a countable (and even two-element for Casino) set of possible moves.

Corollary 18. *If a set $P \subseteq \mathbb{D}$ satisfies $\lambda^+(P) > 0$ and $h = (\lambda^+(P) + \lambda^-(P))/2$, then Gambler does not win $\Gamma^{\mathbb{Q}}(P, h^{-1})$, while if Casino wins $\Gamma^{\mathbb{Q}}(P, h^{-1})$ then the set P is measurable.*

We now show how this implies that every subset of $\mathbb{D}$ is measurable, assuming the axiom of determinacy. The result itself appeared first in [7], and a modern proof can be found, for example, in [8] or [9]. We start by a brief introduction to determinacy.

We consider games involving two players, usually denoted by I and II, making alternating choices from a fixed countable set X. Player I plays first and chooses $x_0 \in X$, player II responds by $x_1 \in X$, then I once again moves $x_2 \in X$, II selects $x_3 \in X$, and so on ad infinitum. A run of the game produces an infinite sequence $\vec{x} = \langle x_n \rangle_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ of elements of the set X, where the moves of player I are the x_{2n}, and the moves of player II are the x_{2n+1}. A pay-off set $P \subseteq X^{\mathbb{N}}$ is fixed before the game starts. Either player has the full knowledge of this set, and before each of his moves knows all the previous choices (by him and his opponent) in the game. The result is defined as follows: if $\vec{x} \in P$, then II wins the run, otherwise I wins the run. This game is denoted by $G(P)$.

A strategy for one of the players in $G(P)$ is a rule (or function) which tells the player what to pick on the nth turn on the basis of the previous choices. A strategy is a *winning* strategy if the player wins all the games in which he follows the choices of this strategy, regardless of the answers made by the opponent. The game $G(P)$ and the set P are called *determined* if one of the two players (it cannot be both players) has a winning strategy.

Note that the fact that I has a winning strategy in $G(P)$ can be expressed by the formula

$$\exists\, x_0\; \forall\, x_1\; \exists\, x_2\; \forall\, x_3\; \cdots\; (\vec{x} \notin P).$$

(We mean here that the strategy for I selects the existentially quantified values x_{2n} on the basis of all values of x_i for $i < 2n$.) Accordingly, if II has a winning strategy, then this can be expressed as

$$\forall\, x_0\; \exists\, x_1\; \forall\, x_2\; \exists\, x_3\; \cdots\; (\vec{x} \in P).$$

Thus, the statement *the game $G(P)$ is determined* is expressed by the following infinitary form of the law of the excluded middle:

$$\exists\, x_0\; \forall\, x_1\; \exists\, x_2\; \forall\, x_3\; \cdots\; (\vec{x} \notin P) \quad \vee \quad \forall\, x_0\; \exists\, x_1\; \forall\, x_2\; \exists\, x_3\; \cdots\; (\vec{x} \in P).$$

The axiom of determinacy AD consists in the statement that all sets are determined, or, saying it differently, that for any set P the game $G(P)$ is determined. It is known that this is a very powerful axiom; in fact, it contradicts the full axiom of choice AC, but is consistent with the axiom of dependent choices DC. (The latter allows an infinite sequence of choices even in the case when the set from which the next choice should be made depends on the results of previous choices.) To be more precise, the consistency of AD + DC with the axioms of Zermelo-Fraenkel set theory ZF (without the axiom of choice) has been established under the assumption that some other proposition (namely, the statement of existence of infinitely many so-called *Woodin cardinals*) is consistent with ZF, and, in fact these two consistency hypotheses are equivalent. Woodin cardinals belong to the family of *large cardinals* (together with, for example, inaccessible and measurable cardinals, but they are much bigger that the latter two types); see more on this in [10], Ch. 33.

On the other hand, we can prove that fairly simple sets are determined. For instance, Martin [11] showed that all Borel sets $P \subseteq X^{\mathbb{N}}$ are determined. (We recall that the set X is at most countable, for example $X = \mathbb{N}$.)

One can hardly hope that the determinacy theory for games on countable sets X is directly applicable to the analysis of games of the form $\Gamma(P)$ since they do not restrict possible moves of both players by a countable set. Yet it is quite clear that the 'rational' modifications $\Gamma^{\mathbb{Q}}(P)$, where the moves are restricted to a countable set, can be coded as games of the form $G(P')$ for appropriate sets $P' \subseteq \mathbb{N}^{\mathbb{N}}$ (so that each player chooses natural numbers). Thus, AD implies that for each set $P \subseteq \mathbb{D}$ and each value of h, $0 \leqslant h \leqslant \infty$, the game $\Gamma^{\mathbb{Q}}(P, h)$ is determined. Then Corollary 18 provides another proof of the following known result (first established in [7]; see a presentation in Russian in [9]).

Corollary 19. *Under the assumption of the axiom of determinacy* AD *every set $P \subseteq \mathbb{D}$ is Lebesgue measurable.*

§ 7. A 'discrete' modification of the game

Here we introduce yet another, and more substantial, modification of games of the form $\Gamma(P)$. It involves increasing Gambler's initial balance B_0 to some positive integer, possibly a very large one, and requiring Gambler to bet only integer amounts and (to avoid some trivialities) to bet non-zero amounts infinitely often. Clearly this is essentially the same as keeping the initial balance of 1 dollar and allowing to bet only multiples of $1/n$ for a fixed natural n. We use $\Gamma^{\mathbb{Z}}(P, B_0)$ to denote this game.

The results relating to this game are strikingly different from those for $\Gamma(P)$, and we shall see that, in fact, Casino can eventually force Gambler to go bankrupt.

Theorem 20. *If $B_0 \in \mathbb{N}$ and $P \subseteq \mathbb{D}$ satisfies $\lambda^-(P) > 0$, then Casino has a strategy in the game $\Gamma^{\mathbb{Z}}(P, B_0)$ that guarantees $B_{n_0} = 0$ for some n_0 (and then also for all $n > n_0$).*

Proof. The main idea is as follows. In the original version of the game with real-valued bets, if Gambler's optimal strategy calls for a very small stake, say, 1/2 cent, then in the 'discrete' case Gambler can either bet 1 dollar (or more), therefore losing in the quality of the position and balance, or pass with zero stake, which allows Casino to play either digit and hence keep the amount of the next optimal bet small; but by definition eventually Gambler has to bet a non-zero amount and make a loss! This happens, for example, when the relative measure of P in the current domain $\mathbb{D}_{\vec{p}\restriction n}$ is close to 1. After partitioning this domain into two smaller subintervals, the corresponding relative measures will again be close to 1, which implies another small stake. We shall show that, at the cost of a very small loss, Casino can force Gambler to 'visit' positions with relative measures very close to 1 from time to time. Now we come to the details.

We consider the case of a closed set P first. Let ℓ be the least integer greater than $B_0/\lambda(P)$, the initial P-quality. Thus, if Casino follows the P-quality decreasing strategy then the balances will remain less than ℓ and the sequence $\vec{p}$ of Casino's plays will belong to the set P. It suffices to show that Casino has a strategy which, after finitely many turns, reduces this P-quality by a certain fixed amount (for instance, 0.6) since once the quality decreases by this amount, Casino can assume it is starting a new game for the set $P \cap \mathbb{D}_s$ (s being the play of Casino to this point) to decrease the quality again by 0.6 by using the same strategy, and so on.

Here follows the description of one such strategy for Casino.

Fix a very small $\epsilon > 0$; the precise value will be made clear later. There is a finite set $\mathscr{I}$ of pairwise disjoint Baire intervals $\mathbb{D}_s$ such that $P \subseteq U = \bigcup \mathscr{I}$ and $\lambda(U) < \lambda(P) + (\varepsilon/\ell)^2$. Let

$$\mathscr{I}' = \left\{ \mathbb{D}_s \in \mathscr{I} : \lambda_s(P) > 1 - \frac{\varepsilon}{\ell} \right\},$$

so that for the sets $U' = \bigcup \mathscr{I}'$ and $P_1 = P \cap U'$ we obtain $m_1 = \lambda(P_1) \geqslant \lambda(P) - \varepsilon/\ell$.

The first requirement for ε: $\frac{B_0}{\lambda(P)-\varepsilon/\ell} < \ell$.

The second requirement for ε: $\frac{B_0}{\lambda(P)-\varepsilon/\ell} < \frac{B_0}{\lambda(P)} + 0.1$.

The first requirement guarantees that the initial P_1-quality is less than ℓ, and the second guarantees that the initial P_1-quality is within 0.1 from the original P-quality. Casino starts by following the P_1-quality decreasing strategy, so that the balances will remain less than ℓ. By the definition of P_1, after several turns, the finite sequence $\vec{p} \upharpoonright n = s$ of Casino's plays satisfies $\mathbb{D}_s \in \mathscr{I}'$. At this moment, we see that $d = \lambda_s(P) = \lambda_s(P_1) > 1 - \varepsilon/\ell$. Therefore, the corresponding quantities $d_i = \lambda_{s^\frown i}(P) = \lambda_{s^\frown i}(P_1)$, $i = \pm 1$, both satisfy $1 \geqslant d_i > 1 - 2\varepsilon/\ell$. Let B be the current balance at this point (that is, after the last play by Casino in the sequence s) and b the next bet by Gambler, so that $|b| \leqslant B < \ell$. Then the values q, q_{-1}, and q_1 of the P_1-quality for the sequences s, $s^\frown(-1)$, and $s^\frown 1$, respectively, satisfy the relations

$$q = \frac{B}{d}\,, \qquad q_{-1} = \frac{B-b}{d_{-1}}\,, \qquad q_1 = \frac{B+b}{d_1}\,.$$

In this position, the P-quality preserving bet by Gambler will be close to 0. However, by definition the stakes can now be only integers.

Case 1: Gambler bets $b \neq 0$. Then

$$q_{-1} - q_1 = \frac{B(d_1 - d_{-1}) - b(d_{-1} + d_1)}{d_{-1}d_1}\,.$$

However, $|d_1 - d_{-1}| \leqslant 2\varepsilon/\ell$, so $B|d_1 - d_{-1}| \leqslant \ell\frac{2\varepsilon}{\ell} = 2\varepsilon$. On the other hand, $|b|(d_1 + d_{-1}) = |b|2d \geqslant 2d > 2 - 2\varepsilon/\ell$.

The third requirement for ε: $2\varepsilon < 0.1$ and at the same time $2 - 2\varepsilon/\ell > 1.7$.

If this holds, then $|q_{-1} - q_1| > 1.6$, and hence at least one of the two possible new qualities is far away from the current quality. Furthermore, since $d = (d_{-1} + d_1)/2$, it follows that

$$\begin{aligned} |q_{-1} + q_1 - 2q| &= \left| \frac{B(d_1 - d_{-1})^2 + b(d_{-1}^2 - d_1^2)}{2dd_{-1}d_1} \right| \\ &\leqslant \left| \frac{B(d_1 - d_{-1})^2 + |b|\,|d_{-1}^2 - d_1^2|}{2dd_{-1}d_1} \right| \leqslant \ell\left[2\left(\frac{\varepsilon}{\ell}\right)^2 + 2\frac{\varepsilon}{\ell}\right]. \end{aligned}$$

The fourth requirement for ε: $\varepsilon^2/\ell + 2\varepsilon < 0.2$.

If this holds, then $|q_{-1} + q_1 - 2q| < 0.2$. Therefore, one of the quantities q_{-1} and q_1 satisfies the inequality $q_i < q - 0.7$. If $q_1 < q - 0.7$, then Casino plays 1, otherwise

Casino plays -1. Thus, the P_1-quality has been decreased at least by 0.7. On the other hand, the final P_1-quality is equal to the final P-quality, since $P\cap\mathbb{D}_s=P_1\cap\mathbb{D}_s$, while the initial P_1-quality is not greater than the initial P-quality plus 0.1. Hence, in general, we have succeeded in decreasing the P-quality by at least 0.6.

Case 2: Gambler bets $b=0$.

In this case Casino merely 'waits' by playing $p_n=\pm1$, so that the relative measure $\lambda_{s^\frown p_n}(P_1)\geqslant\lambda_s(P_1)$, and therefore, the quality does not increase. Casino continues to play in this manner until the next turn of Gambler making a non-zero bet. At this point, as outlined above in Case 1, Casino can reduce the P-quality by at least 0.6.

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