ON THE EXTENSION PRINCIPLE IN INTERNAL SET THEORY

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Introduction

The internal set theory (IST) expounded by Nelson [1, 2] is considered now by specialists as one of the most successful axiomatizations of nonstandard methods. Not only it has found applications to various fields of nonstandard mathematics presenting tools simple, convenient and, at the same time, rather powerful for formalization of the "infinitesimal" reasoning, but also it becomes a subject of in-depth studies for its own sake (see [3–8]).

As opposed to other axiomatizations of nonstandard methods, Nelson's theory is distinguished by the fact that it operates only with the two types of sets: *standard* and *internal*, whereas, for example, the systems proposed by Hrbaček [9], Kawai [10], Henson and Keisler [11] (see a survey of the subject in [12]) are in a more complete agreement with the "constructive" version of nonstandard approach consisting in the direct construction of enlargements for standard structures (see [8, 13, 14]) and allow of the sets of the third type: *external* sets.

The structure of IST, being clearer in this respect, simplifies the process of distinguishing genuinely topical logical questions, stating and solving the classical set theoretic problems of consistency, independence, and undecidability.

IST results from adjoining the unary predicate st of standardness to the language of Zermelo—Fraenkel set theory (ZFC) and three new axioms (idealization, standardization, and transfer; see them below) to the list of the axioms of ZFC, these new axioms regulating the properties of sets connected with the notion of standardness.

Together with the above postulates, the following *extension principle* is sometimes used in the studies connected with IST (cf. [6–8]):

(E) *If X, Y is a pair of standard sets and $\Phi(x,y)$ is an* st-$\in$-*formula such that, for each standard $x \in X$, there is a $y \in Y$ satisfying $\Phi(x,y)$, then there exists a function $\tilde{y}$ such that $\tilde{y}(x)$ is defined and satisfies $\Phi(x,\tilde{y}(x))$ for every standard $x \in X$. Formally,*

$$\forall^{\text{st}} x \in X\ \exists y \in Y\ \Phi(x,y) \rightarrow \exists \tilde{y}\ \forall^{\text{st}} x \in X\ [\Phi(x,\tilde{y}(x))\ \ \&\ \ \tilde{y}(x) \in Y].$$

It is admissible that the formula Φ contains the standardness predicate as well as free variables other than x and y (in interpretation they are replaced by arbitrary sets, *parameters*).

We will write $\forall^{\text{st}} x \in X\ \Psi(x)$ to denote $\forall x(\text{st}\, x \rightarrow \Psi(x))$, and $\exists^{\text{st}}$ will be understood in a similar sense. Note that the quantifiers $\exists^{\text{st}}$, $\forall^{\text{st}}$ are conveniently called *external*, while the quantifiers $\exists$ and $\forall$ without the superscript st are called *internal.* The words "external" and "internal" are by no means connected with a position of a quantifier in a formula. Similarly, any st-$\in$-formula not containing the predicate st is referred to as *internal*, while a formula containing this predicate is called *external.*

It should be emphasized that (E) becomes false in IST if we replace $\exists^{\text{st}} x$ by $\exists x$ on the left-hand and right-hand sides.

In [2] it was proved that (E) is a theorem of IST in case each external quantifier (in the above-indicated sense) of the formula Φ is of the form $\exists^{\text{st}} z \in Z$ or $\forall^{\text{st}} z \in Z$, where Z is some standard set possibly depending upon a quantifier, and there are no occurrences of the standardness predicate st other than those with the quantifiers $\exists^{\text{st}}$ and $\forall^{\text{st}}$ of the type indicated (in what follows such formulas will be called ext-*restricted*). This result is quite sufficient to guarantee legitimacy for the applications of (E) known to the author; nevertheless, it does not answer the natural question about the general status of the extension principle in IST. The following theorem proposes a solution:

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Theorem 1. (E) *is undecidable in* IST.

The proof of this theorem forms the bulk of the exposition. First, constructing an inner model, we will prove that (E) is consistent with the axioms of IST. (There is another (longer) way of proving consistency for (E) by checking it in the model employed by Nelson in [1].) Second, we will construct a model of IST in which (E) fails for $X = Y = \mathbb{N}$ (the set of integers) and for a certain explicitly-given formula Φ. In our example, the formula Φ negating (E) is of type Π_2^{st}, i.e. of the form $\forall^{\text{st}} a\ \exists^{\text{st}} b\, \varphi(x, y, a, b)$, where φ is an internal formula which, to our regret, must contain some nonstandard set as a parameter. Thus the problem to construct a model of IST in which (E) will be false for a formula without parameters (at least having only standard parameters) remains unsolved.

The situation changes, however, if we omit the requirement that y belong to a given set; i.e., if we consider the following hypothesis:

(E1) $\forall^{\text{st}} x \in X\ \exists y\ \Phi(x, y) \to \exists \tilde{y}\ \forall^{\text{st}} x \in X\ \Phi(x, \tilde{y}(x))$.

It is easy to see that (E1) implies (E); indeed, insert the formula $\Phi(x, y)\ \&\ y \in Y$ instead of $\Phi(x, y)$ in (E1).

Theorem 2. *The hypothesis* (E1) *is not deducible in* IST *even for formulas* Φ *without parameters.*

More precisely, we assert that there exists a formula $\Phi(x, y)$ without parameters and of type Π_2^{st} for which the negation of (E1) is consistent with IST.

The article is arranged as follows. In §1 we construct an inner model of IST in which (E) and (E1) hold. This will prove consistency with IST axioms for (E) and (E1) (clearly, the consistency of (E) ensues from that of (E1)). Then, in §2, we prove an important technical theorem on expressibility by an appropriate external formula of validity for any internal formula with standard parameters. This theorem is used in §3 for constructing a model of IST in which the negations of (E) and (E1) hold for some explicitly-defined matrix formulas Φ. This completes the proof of Theorems 1 and 2.

In fact, we shall prove that the following weaker versions of (E) and (E_1):

(E!) $\forall^{\text{st}} x \in X\ \exists! y \in Y\ \Phi(x, y) \to \exists \tilde{y}\ \forall^{\text{st}} x \in X\ [\Phi(x, \tilde{y}(x)) \& \tilde{y}(x) \in Y]$,

(E1!) $\forall^{\text{st}} x \in X\ \exists! y\ \Phi(x, y) \to \exists \tilde{y}\ \forall^{\text{st}} x \in X\ \Phi(x, \tilde{y}(x))$,

in which the uniqueness requirement for y is added, are not deducible in IST.

§1. Consistency of the Extension Principle

We recall the statements of the additional IST axioms:

Idealization (I). For every internal formula $\Phi(x, a)$,

$$\forall^{\text{st fin}} A\ \exists x\ \forall a \in A\ \Phi(x, a) \leftrightarrow \exists x\ \forall^{\text{st}} a\ \Phi(x, a).$$

Standardization (S). For every st-$\in$-formula $\Phi(x)$ and any standard set X,

$$\exists^{\text{st}} Y\ \forall^{\text{st}} x\ [x \in Y \leftrightarrow x \in X\ \&\ \Phi(x)].$$

Transfer (T). For every internal formula $\Phi(x)$ with standard parameters,

$$\exists x\ \Phi(x) \leftrightarrow \exists^{\text{st}} x\ \Phi(x).$$

Thus, IST is ZFC plus (I) plus (S) plus (T).

Consistency with the axioms of IST for (E) and (E1) will be proved in the following form:

Theorem 3 (IST). *Let* $\varkappa$ *be a standard infinite cardinal such that* $\mathbb{V}_\varkappa$ *is a model of* ZFC. *Then* $\mathbb{V}_\varkappa$ *is a model of* IST *in which* (E) *and* (E1) *hold.*

Here we denote by $\mathbb{V}_\varkappa$ the $\varkappa$th level of the von Neumann set hierarchy [15]. Thus, $\mathbb{V}_0 = \varnothing$, $\mathbb{V}_{\alpha+1} = \mathcal{P}(\mathbb{V}_\alpha) = \{X : X \subseteq \mathbb{V}_\alpha\}$ for any ordinal α, and $\mathbb{V}_\lambda = \cup_{\alpha<\lambda} \mathbb{V}_\alpha$ for any limit ordinal λ. Finally, $\mathbb{V} = \cup_{\alpha \in \text{Ord}} \mathbb{V}_\alpha$ is the universe of all sets.

Proof. We put $V = \mathbb{V}_\varkappa$. By a *V-restricted* formula we will mean any st-$\in$-formula whose quantifiers (external and internal) are restricted to the set V. It is clear that any assertion about the validity of internal formulas in V is expressible by a V-restricted formula; this fact gives grounds for our proof of the theorem.

To check the transfer principle in V, i.e. the implication

$$\exists x \in V\ \Phi(x) \to \exists^{\text{st}} x \in V\ \Phi(x),$$

for an internal V-restricted formula Φ with standard parameters (certainly, only parameters lying in V are of interest here; the result, however, remains true for arbitrary standard parameters), it suffices to apply the transfer principle of IST to the formula $\Phi(x)\ \&\ x \in V$ (which is internal and has standard parameters). In precisely the same manner we can validate principles (S) and (I) in V; the relevant simple consideration is left to the reader.

It remains to check that (E1) holds in V, i. e. to verify the formula

$$\forall^{\text{st}} x \in X\ \exists y \in V\ \Phi(x, y) \to \exists \tilde{y} \in V\ \forall^{\text{st}} x \in X\ \Phi(x, \tilde{y}(x)),$$

where $X \in V$ is a standard set and Φ is an arbitrary V-restricted st-$\in$-formula. However, the specific feature of V-restricted formulas (in fact, of all st-$\in$-formulas in which every external quantifier (in the sense indicated) is restricted to a suitable standard set) consists in the fact that, for them, (E) and (E1) become theorems of IST; this was proved in [2]. Consequently, there exists a function $\tilde{y}$ for which

$$\forall^{\text{st}} x \in X\ [\tilde{y}(x) \in V\ \&\ \Phi(x, \tilde{y}(x))]. \qquad (*)$$

Moreover, it is required that $\tilde{y} \in V$. To satisfy this additional condition, we let H be a finite set containing all the standard elements of V (the existence of such a set H was established in [1]). We put

$$D = H \cap V \cap \operatorname{dom} \tilde{y}; \quad E = \{x \in D : \tilde{y}(x) \in V\}; \quad \tilde{f} = \tilde{y} \mid E.$$

Clearly, $\tilde{f}$ is a function with finite domain of definition $E \subseteq V$ and range in V; i.e., $\tilde{f} \in V$. Moreover, the needed property $(*)$ passes from $\tilde{y}$ to $\tilde{f}$. □

In the results like Theorem 3 (and, naturally, Theorems 1 and 2), a particular emphasis is attracted to the tools used in proving (say, for constructing a required model). In the above proof we have used as an assumption the existence of a cardinal $\varkappa$ such that $\mathbb{V}_\varkappa$ is a model of ZFC; this assumption certainly falls outside the scope of either ZFC or IST (since the two theories are equiconsistent) and, in fact, it is even stronger than the hypothesis Cons ZFC of the formal consistency of ZFC. Thus, Theorem 3 (in the very form it was straightforwardly established) does not give a proof of consistency with IST for (E) and (E1) in the most desirable form:

$$\text{Cons IST} \to \text{Cons IST} + (\text{E}) + (\text{E1}).$$

Consider, however, the theory $\text{IST}_\varkappa$ that is obtained by enriching the language of IST with the additional constant $\varkappa$ and appending to the list of the axioms of IST the axiom "$\varkappa$ is a standard cardinal" together with the collection of axioms of the form "A is valid in $\mathbb{V}_\varkappa$", where A ranges over all the axioms of ZFC. Denote the corresponding extension of ZFC by $\text{ZFC}_\varkappa$. Of course, $\text{ZFC}_\varkappa$ is not the same as the result of adding to ZFC the single axiom that $\mathbb{V}_\varkappa$ is a model of ZFC (the latter version gives a stronger extension of ZFC).

It is easy to check that $\text{IST}_\varkappa$ is equiconsistent with IST (and is a conservative extension of the latter); however, $\text{IST}_\varkappa$ is strong enough to ensure provability in this theory of any axiom of IST as well as the validity of (E) and (E1) in $\mathbb{V}_\varkappa$. The needed proof (in $\text{IST}_\varkappa$) practically mimics that of Theorem 3 and, therefore, is omitted. □

§2. Truth Verification for Internal Formulas

The purpose of the present section is to prove a theorem about truth verification for internal formulas with standard parameters by means of some external formula. This result partly interesting by itself is an important technical tool which will be used in §3 in proving independence for (E) and (E1). Before stating the theorem exactly, we expose necessary definitions connected with encoding the $\in$-language by means of finite sequences of symbols related to the logical symbols and sets used as parameters.

For the sake of simplicity, we assume that $\in$-formulas are written only with the help of the logical symbols $\neg$, $\&$, $\exists$, $\in$, $=$; the brackets (and); the variables v and v_i, $i \in \mathbb{N}$; and, finally, the parameters, i.e. arbitrary sets which can replace free variables. The other logical connectives are well known to be expressible in terms of $\neg$, $\&$, and $\exists$.

Given a $\in$-formula φ, the *translation* of it is understood to be the string $\ulcorner\varphi\urcorner$ obtained from φ by replacing

the symbols $\neg$, $\&$, $\exists$, $\in$, $=$, (,) with the integers 0, 1, 2, 3, 4, 5, 6;

each of the variables v_i with $8+i$ and v with 7;

each parameter p ($p \in \mathbb{V}$) with the ordered pair $\langle 0, p\rangle$.

Thus, $\ulcorner\varphi\urcorner$ is a finite string whose entries are sets of a special type. All these strings compose the collection

$$\text{Form} = \{\ulcorner\varphi\urcorner : \varphi \text{ is a (well formed) } \in\text{-formula}\}$$

in which, for any X, we can distinguish

$$\text{Form}_X = \{\ulcorner\varphi\urcorner \in \text{Form} : \text{all parameters of } \varphi \text{ belong to } X\}.$$

Theorem 4. *There exists an* st-$\in$*-formula* $\tau(x)$ *such that, for any internal formula* $\Phi(x_1,\ldots,x_n)$*, the following relation holds in* IST:

$$\forall^{\mathrm{st}} x_1 \ldots \forall^{\mathrm{st}} x_n\ [\Phi(x_1,\ldots,x_n) \leftrightarrow \tau(\ulcorner\Phi(x_1,\ldots,x_n)\urcorner)].$$

Proof. We need one more definition. We will say that a formula ψ is *subordinate* to a formula φ if ψ is such a subformula of φ in which some (possibly, all or none) free variables are replaced by parameters. In particular, φ itself is subordinate to φ. We put

$$\text{Form}[\varphi] = \{\ulcorner\psi\urcorner : \psi \text{ is subordinate to } \varphi\}; \quad \text{Form}_X[\varphi] = \text{Form}_X \cap \text{Form}[\varphi].$$

For instance, $\ulcorner\varphi(p)\urcorner \in \text{Form}_X[\exists v\ \varphi(v)]$ for any $p \in X$.

Finally, we distinguish the translations of closed formulas by setting

$$\text{CForm} = \{\ulcorner\varphi\urcorner \in \text{Form} : \varphi \text{ is a closed formula}\},$$

and define CForm_X, $\text{CForm}[\varphi]$, and $\text{CForm}_X[\varphi]$ similarly.

Note that the translation $\ulcorner\varphi\urcorner$ is standard (viewed as a finite string) if and only if φ has only standard parameters and the (natural) number of logical symbols in the expression of φ is also standard.

Now we give our key definition. We denote by $\text{Sat}(T)$ the conjunction of the following five formulas in st-$\in$-language:

$$T \subseteq \text{CForm} \quad (T \text{ consists of translations of closed formulas}); \tag{1}$$

$$\forall^{\mathrm{st}} p\ \forall^{\mathrm{st}} q\ [(\ulcorner p = q\urcorner \in T \leftrightarrow p = q)\ \ \&\ \ (\ulcorner p \in q\urcorner \in T \leftrightarrow p \in q)]; \tag{2}$$

$$\forall^{\mathrm{st}} \ulcorner\varphi\urcorner\ \forall^{\mathrm{st}} \ulcorner\psi\urcorner\ [\ulcorner\varphi\ \&\ \psi\urcorner \in T \leftrightarrow (\ulcorner\varphi\urcorner \in T\ \ \&\ \ \ulcorner\psi\urcorner \in T)]; \tag{3}$$

$$\forall^{\mathrm{st}} \ulcorner\varphi\urcorner \in T\ \forall^{\mathrm{st}} \ulcorner\psi\urcorner \in \text{CForm}[\varphi]\ (\ulcorner\neg\psi\urcorner \in T \leftrightarrow \ulcorner\psi\urcorner \notin T); \tag{4}$$

$$\forall^{\mathrm{st}} \ulcorner\varphi(v)\urcorner\ [\ulcorner\exists v \varphi(v)\urcorner \in T \leftrightarrow \exists^{\mathrm{st}} p\ (\ulcorner\varphi(p)\urcorner \in T)]. \tag{5}$$

Thus, each set T satisfying $\text{Sat}(T)$ has a structure adapted to determining the truth within the universe $\mathbb{S}$ of standard sets by means of T. Namely,

Lemma 5 (IST). *In order that a given closed* $\in$*-formula* φ *be true in* $\mathbb{S}$ (*or, which is tantamount by transfer, in the universe* $\mathbb{V}$ *of all sets*), *it is necessary and sufficient that either of the following two equivalent conditions be satisfied*:

$$\exists T\ [\mathrm{Sat}(T)\ \ \&\ \ \ulcorner\varphi\urcorner \in T]\ \textit{and}\ \forall T\ [\mathrm{Sat}(T) \to \ulcorner\neg\varphi\urcorner \notin T].$$

More precisely, for an arbitrary $\in$-formula $\varphi(\mathbf{v})$ with free variables $\mathbf{v} = v_1, \dots, v_n$, the next relation holds in IST:

$$\forall^{\mathrm{st}}x_1 \dots \forall^{\mathrm{st}}x_n\ [\varphi(x_1, \dots, x_n) \leftrightarrow \exists T\ [\mathrm{Sat}(T)\ \ \&\ \ \ulcorner\varphi(x_1, \dots, x_n)\urcorner \in T]$$

$$\leftrightarrow \forall T\ [\mathrm{Sat}(T) \to \ulcorner\neg\varphi(x_1, \dots, x_n)\urcorner \notin T]].$$

The proof of the lemma should be understood in the sense just indicated.

Proof. We prove two auxiliary claims from which Lemma 5 ensues immediately. □

Claim 1. *For every closed* $\in$*-formula* φ *with standard parameters, there exists a set* T *satisfying* $\mathrm{Sat}(T)$ *and containing at least one of the translations* $\ulcorner\varphi\urcorner$ *and* $\ulcorner\neg\varphi\urcorner$.

Proof. Replace all the parameters in φ with free variables. Let $\varphi(v_1, \dots, v_n)$ be the resulting formula and let $\varphi_i(v_{i_1}, \dots, v_{i_{n(i)}})$, $1 \le i \le m$, $i_n \in \mathbb{N}$, be the list of all its subformulas (φ inclusively). Take a set H which contains all standard sets (cf. [1]) and define

$$T_i = \{\ulcorner\varphi_i(x_1, \dots, x_{n(i)})\urcorner : x_1, \dots, x_{n(i)} \in H\ \ \&\ \ \varphi_i(x_1, \dots, x_{n(i)})\}$$

$$\cup \{\ulcorner\neg\varphi_i(x_1, \dots, x_{n(i)})\urcorner : x_1, \dots, x_{n(i)} \in H\ \ \&\ \ \neg\varphi_i(x_1, \dots, x_{n(i)})\}.$$

The set $T = \cup_{1 \le i \le m} T_i$ is the one sought. □

Claim 2. *For any closed* $\in$*-formula* φ *with standard parameters, if* $\ulcorner\varphi\urcorner \in T$ *and* $\mathrm{Sat}(T)$ *holds then* φ *is true in* $\mathbb{S}$ *as well as in* $\mathbb{V}$.

Proof. We proceed by induction on the number of symbols in the string $\ulcorner\varphi\urcorner$. The base of induction (i.e. the formulas $x \in y$ and $x = y$ for standard x and y) is guaranteed by formula (2) of the definition of Sat, and the induction steps, by formulas (3)–(5). The only nontrivial case is the step $\neg$ that is considered separately. Thus, assume that $\ulcorner\neg\varphi\urcorner \in T$; we are to prove that φ is false. Observe immediately that $\ulcorner\varphi\urcorner \notin T$ in view of (4).

Case 1. φ is an atomic formula $x \in y$ or $x = y$ with x and y standard. From (2) it ensues that $x \notin y$ (respectively, $x \ne y$); for, $\ulcorner\varphi\urcorner \notin T$. Hence, φ is false.

Case 2. φ equals $\psi\ \ \&\ \ \chi$. At least one of the translations $\ulcorner\psi\urcorner$ or $\ulcorner\chi\urcorner$ does not belong to T by (3); say, $\ulcorner\psi\urcorner \notin T$. Then $\ulcorner\neg\psi\urcorner \in T$ by (4). So $\neg\psi$ is true by the induction hypothesis. Hence, ψ is false.

Case 3. φ equals $\neg\psi$. Then $\ulcorner\psi\urcorner \in T$ by (4), since $\ulcorner\varphi\urcorner = \ulcorner\neg\psi\urcorner \notin T$. Consequently, ψ is true by the induction hypothesis; therefore, φ is false.

Case 4. φ equals $\exists v\ \psi(v)$. We are to prove that $\psi(x)$ is false if x is standard. From (5) we obtain $\ulcorner\psi(x)\urcorner \notin T$ (for, $\ulcorner\varphi\urcorner \notin T$). Thus, $\ulcorner\neg\psi(x)\urcorner \in T$ by (4); i.e., $\psi(x)$ is false by the induction hypothesis.

This completes the proofs of Lemma 5 and Theorem 4. Indeed, in the role of τ we can take either of the two formulas of Lemma 5, $\exists T\ [\mathrm{Sat}(T)\ \ \&\ \ \ulcorner\varphi\urcorner \in T]$ or $\forall T\ [\mathrm{Sat}(T) \to \ulcorner\neg\varphi\urcorner \notin T]$. □

§3. Independence of the Extension Principle

The independence of a proposition (in our case, (E) or (E1)) means that it is impossible to prove this proposition in the theory under consideration, i.e., here, in IST. We shall accomplish such a proof by way of constructing a model of IST in which (E) and (E1) fail; moreover, (E1) is violated for some formula Φ without parameters.

Our constructing such a model is carried out within ZFC under some additional technical suppositions made for the sake of convenience. We will use the technique based on the adequate ultralimit construction of [1]; however, the specific features of the problem need not only the use of a special choice for the initial model of ZFC but also a special way of its nonstandardly enlarging, the choice and enlargement slightly different from those of [1].

3.1. An initial model. We will assume, making our consideration in ZFC, the existence of a cardinal ϑ such that $\mathbb{V}_\vartheta$ is a model of ZFC. In point of fact, our assumption falling out the scope of ZFC and taken only for the sake of convenience can be eliminated in approximately the same way as before in §1, i.e., by considering a suitable extension of ZFC.

Now, let ϑ be a cardinal satisfying the property indicated; so that $\mathbb{V}_\vartheta$ is a model of ZFC. It is convenient to assume ϑ to be the least cardinal of the kind.

One more (final) assumption is that we will presuppose (in our constructing a model of IST in ZFC) the axiom of constructibility $\mathbb{V} = \mathbb{L}$. A consequence, of this axiom (that is consistent with ZFC), essential for us is a well ordering of the universe of all sets by means of a certain explicitly-defined $\in$-formula possessing the property that the restriction of the order onto an arbitrary set of the form $\mathbb{V}_\vartheta$, where ϑ is a cardinal, is definable in $\mathbb{V}_\vartheta$ and well orders $\mathbb{V}_\vartheta$ with order type ϑ (cf. [15]).

We now fix a "natural" enumeration $\varphi_k(v_1, \dots, v_{m(k)})$, $k \in \mathbb{N}$, of all $\in$-formulas without parameters and with a definite indication of the list of free variables. It is easy to check that, for any n, there exists a cardinal $\varkappa < \vartheta$ such that $\mathbb{V}_\varkappa$ is an elementary submodel of $\mathbb{V}_\vartheta$ of all formulas $\varphi_k(p_1, \dots, p_{m(k)})$, $k \le n$, $p_i \in \mathbb{V}_\varkappa$. Let $\varkappa_n$ denote the least of these cardinals $\varkappa$. Clearly, $\varkappa_n \le \varkappa_{n+1}$ for all n, $\varkappa = \sup\{\,\varkappa_n : n \in \mathbb{N}\,\}$ is a cardinal, and $\mathbb{V}_\varkappa$ is an elementary submodel of $\mathbb{V}_\vartheta$ of all $\in$-formulas with parameters in $\mathbb{V}_\varkappa$; consequently, $\mathbb{V}_\varkappa$ is a model of ZFC. Therefore, $\varkappa = \vartheta$.

It is the set $M = \mathbb{V}_\vartheta$ that we shall take as the initial model of our constructing the needed nonstandard enlargement. Our way of enlargment is connected with the use of *definable* functions for constructing an ultrapower. So we recall necessary definitions.

First of all, we will introduce a second denotation for $\mathbb{V}_\vartheta$ by designating $V = \mathbb{V}_\vartheta$; that this additional denotation is desirable can be explained by the fact that $\mathbb{V}_\vartheta$ seems to play two different roles in our construction: it is the initial model and besides the "universe" for analyzing definability.

Observe that $\varkappa_n \in V$ for all n. Indeed, it suffices to check that $\varkappa_n < \vartheta$ for any n. Suppose the contrary; i.e., that $\varkappa_n = \vartheta$ for all $n \ge n_0$, $n_0 \in \mathbb{N}$. Then, since V is a model of ZFC, for $n = n_0 + 1$ there is a cardinal $\varkappa$, $\varkappa \in V$ (and, consequently, $\varkappa < \vartheta$), such that $\mathbb{V}_\varkappa$ is an elementary submodel of V of all the formulas φ_k, $k \le n$. As a result, $\varkappa_n \le \varkappa < \vartheta$. A contradiction.

Let $\mathrm{Def}(V)$ denote the collection of all the sets $X \subseteq V$ that are definable in V; i.e., $X \in \mathrm{Def}(V)$ if and only if

$$X = \{\, p \in V : \varphi^V(p) \,\} = \{\, p \in V : \varphi(p) \text{ is true in } V \,\}$$

for some $\in$-formula φ with parameters in V and the single free variable p, where φ^V denotes the relativization of φ to V, i.e. the result of replacement of each of the quantifiers $\exists z$ and $\forall z$ in φ by $\exists z \in V$ and $\forall z \in V$ respectively.

Lemma 6. *The sequence* $\langle \varkappa_n : n \in \mathbb{N} \rangle$ *does not belong to* $\mathrm{Def}(V)$.

Proof. Assume to the contrary that there exists a $\in$-formula $\varphi(n, \varkappa)$, with parameters in V, satisfying the relation

$$\forall n \in \mathbb{N}\ \forall \varkappa \in \mathbb{V}\ [\varkappa = \varkappa_n \leftrightarrow \varphi^V(n, \varkappa)].$$

Then there exists an integer n such that all the parameters of φ belong to $\mathbb{V}_{\varkappa_n}$ and, furthermore, $\mathbb{V}_{\varkappa_n}$ is an elementary submodel of V of the formulas $\varphi(\nu, \varkappa)$ (with free variables ν and $\varkappa$) and $\exists \varkappa \varphi(\nu, \varkappa)$ (with the free variable ν). The formula $\exists \varkappa \varphi(n, \varkappa)$ is true in V (to observe this, take $\varkappa = \varkappa_n$) and, consequently, in $\mathbb{V}_{\varkappa_n}$ as well. Hence, $\varphi(n, \varkappa)$ holds in $\mathbb{V}_{\varkappa_n}$ and thus in V for some $\varkappa \in \mathbb{V}_{\varkappa_n}$. However, the last is possible only in the case $\varkappa = \varkappa_n$. Therefore, $\varkappa_n \in \mathbb{V}_{\varkappa_n}$, a contradiction. □

In point of fact, the sequence of the cardinals $\varkappa_n$ will be of principal value as grounds of our constructing the needed counterexample with (E) and (E1) violated. Our main idea is to construct a nonstandard enlargement of M by using only functions of $\mathrm{Def}(V)$. This will guarantee that the map

$k \mapsto \varkappa_k$ will be out of our enlargement, whereas Theorem 5 will ensure that the this map is definable in the extension by means of an appropriate (external) formula.

Now we fill in details.

3.2. An index set and ultrafilter. As the index set we take

$$I = \mathcal{P}^{\text{fin}}(M) = \{\, i \subseteq M : i \text{ is finite} \,\};$$

it is clear that $I \in \mathrm{Def}(V)$. The desired ultrafilter is given by the following

Theorem 7. *There exists an ultrafilter U over I satisfying the following two properties:*

(a) *if $a \in M$ then $\{\, i \in I : a \in i \,\} \in U$;*

(b) *if $P \subseteq I \times M$, $P \in \mathrm{Def}(V)$ then the set $\{\, p \in M$: the set $I_p = \{\, i : \langle i, p\rangle \in P \,\}$ is in $U \,\}$ belongs to* $\mathrm{Def}(V)$.

Proof. Construction of the desired ultrafilter U will be accomplished in three steps.

1. Define U_0 to be the collection of all sets of the type $\{\, i \in I : a_1, \ldots, a_n \in i \,\}$, where $a_1 \ldots, a_n \in M$. The family U_0 clearly possesses the *finite intersection property* (FIP) which says that the intersection of every finite subfamily of sets of U_0 is nonempty.

2. Fix an enumeration $\chi_k(i,p)$, $k \geq 1$, of all $\in$-formulas without parameters and with two free variables. By the assumption made above, there exists a well ordered set $V = \mathbb{V}_\vartheta$ which is definable in $\mathbb{V}$ and whose order type is ϑ. Denote by p_α $(\alpha < \vartheta)$ the αth element of V with respect to this order. Then the sequence $\langle p_\alpha : \alpha < \vartheta\rangle$ belongs to $\mathrm{Def}(V)$. We define

$$A_k(\alpha) = \{\, i \in I : \chi_k(i, p_\alpha) \text{ is true in } V \,\} \text{ and } C_k(\alpha) = I \setminus A_k(\alpha).$$

Now by induction on k it is not hard to construct a collection of sets $T_k \subseteq \vartheta$, $T_k \in \mathrm{Def}(V)$, such that, with

$$U_k = \{\, A_k(\alpha) : \alpha \in T_k \,\} \cup \{\, C_k(\alpha) : \alpha \in \vartheta \setminus T_k \,\},$$
$$U_{k\gamma} = \{\, A_k(\alpha) : \alpha \in T_k \cap \gamma \,\} \cup \{\, C_k(\alpha) : \alpha \in \gamma \setminus T_k \,\},$$

the family $U_0 \cup \cdots \cup U_{k-1} \cup U_{k\gamma}$ possesses FIP for all $k \geq 1$ and $\gamma < \vartheta$. (To this end, it suffices to take into account the trivial fact that, of two mutually complementary sets, at least one can be adjoint to any family possessing FIP so as to obtain a family also possessing FIP.)

3. We define $U_\infty = \cup_{k \in \mathbb{N}} U_k$ (apparently, the family U_∞ possesses FIP) and enlarge U_∞ to an ultrafilter U over I arbitrarily. It is easily seen that the ultrafilter obtained has the required properties. □

3.3. The quantifier "there exist sufficiently many." The use of properties (a) and (b) of the ultrafilter U constructed is radically simplified in the framework of the formalism of generalized quantifiers. Define the new quantifier $\mathbf{Q} = \mathbf{Q}_U$ by

$$\mathbf{Q}i\ \varphi(i) \text{ if and only if } \{\, i \in I : \varphi(i) \text{ is true in } M \,\} \in U.$$

The following properties of this additional quantifier ensues from the properties (a) and (b) of U and general properties of ultrafilters.

(Q1) if $p \in M$ then $\mathbf{Q}i\ (p \in i)$;

(Q2) if $P \subseteq I \times M$ and $P \in \mathrm{Def}(V)$, then $\{p \in M : \mathbf{Q}i\ (\langle i,p\rangle \in P)\} \in \mathrm{Def}(V)$; i.e., the class $\mathrm{Def}(V)$ is closed under the action of $\mathbf{Q}$;

(Q3) if $\forall i\ [\varphi(i) \to \psi(i)]$ then $\mathbf{Q}i\varphi(i) \to \mathbf{Q}i\psi(i)$;

(Q4) $\mathbf{Q}i\ \varphi(i)\ \&\ \mathbf{Q}i\ \psi(i) \leftrightarrow \mathbf{Q}i\ [\varphi(i)\ \&\ \psi(i)]$;

(Q5) $\mathbf{Q}i\ \neg\varphi(i) \leftrightarrow \neg\ \mathbf{Q}i\ \varphi(i)$;

(Q6) if i is not a free variable of φ then $\varphi \leftrightarrow \mathbf{Q}i\ \varphi$;

(Q7) $\forall i\ \varphi(i) \to \mathbf{Q}i\ \varphi(i) \to \exists i\ \varphi(i)$.

3.4. The construction of a nonstandard enlargement. Let $r \geq 1$. We put

$$I^r = I \times I \times \cdots \times I \qquad (r \text{ times}),$$

$$M^r = \{ f : f \text{ maps } I^r \text{ into } M,\ f \in \mathrm{Def}(V) \};$$

and let $I^0 = \{ 0 \}$ and $M^0 = \{ \langle 0, p \rangle : p \in M \}$. Finally, we define ${}^*M = \bigcup_{r \geq 0} M^r$. For $f \in {}^*M$, we let $r(f)$ denote the unique integer r for which $f \in M^r$.

Further, if $f \in {}^*M$, $q \geq r = r(f)$, and $\mathbf{i} = \langle i_1, \ldots, i_r, \ldots, i_q \rangle \in I^q$, then we define $f[\mathbf{i}] = f(i_1, \ldots, i_r)$. In particular, $f[\mathbf{i}] = f(\mathbf{i})$ for $r = q$. We also define $f[\mathbf{i}] = p$ for $f = \langle 0, p \rangle \in M^0$.

We introduce the binary relations ${}^*\in$ and ${}^*=$ that make ${}^*M$ into a $\in$-structure as follows: If $r = \max\{r(f), r(g)\}$ then

$$f \ {}^*\!\in g \text{ if and only if } \mathbf{Q}i_r\ \mathbf{Q}i_{r-1} \ldots \mathbf{Q}i_1\ (f[\mathbf{i}] \in g[\mathbf{i}]),$$

$$f \ {}^*\!= g \text{ if and only if } \mathbf{Q}i_r\ \mathbf{Q}i_{r-1} \ldots \mathbf{Q}i_1\ (f[\mathbf{i}] = g[\mathbf{i}]),$$

where, naturally, $\mathbf{i} = i_1, \ldots, i_r$.

If $s \in M$ then we define ${}^*s = \langle 0, s \rangle$; ${}^*s \in M^0$.

Finally, we define the predicate of standardness ${}^*\mathrm{st}$ by setting ${}^*\mathrm{st} f$ in ${}^*M$ if and only if there is an $s \in M$ such that $f \ {}^*\!= s$. Thus, all elements of M^0 are standard in ${}^*M$ and there are no other standard elements in ${}^*M$.

Now, for any st-$\in$-formula with parameters in ${}^*M$, we can determine whether it is true or false in ${}^*M$ by respectively replacing the symbols $=$, $\in$, and st with the symbols ${}^*=$, ${}^*\in$, and ${}^*\mathrm{st}$.

For any formula Φ with parameters in ${}^*M$, we define $r(\Phi) = \max\{ r(f) : f \text{ occurs in } \Phi \}$. In case $r \geq r(\Phi)$ and $\mathbf{i} \in I^r$, we let $\Phi[\mathbf{i}]$ denote the formula obtained from Φ by replacing each parameter f with $f[\mathbf{i}]$. Thus, $\Phi[\mathbf{i}]$ is a formula with parameters in M.

Theorem 8 (the Loš theorem). *Let Φ be a closed $\in$-formula with parameters in ${}^*M$ and suppose that $r \geq r(\Phi)$. Then*

$$\Phi \textit{ is true in } {}^*M \leftrightarrow \mathbf{Q}i_r \ldots \mathbf{Q}i_1 \quad (\Phi[i_1, \ldots, i_r] \textit{ is true in } M).$$

Proof. For atomic formulas, the claim follows immediately from the definition. Now we proceed by induction on the number of logical connectives and, in making the induction steps, we can confine ourselves with considering only the symbols $\neg$, & , and $\exists$.

The induction steps with & and $\neg$ cause no difficulties: the required result ensues immediately from the properties (Q4)–(Q6) of the quantifier $\mathbf{Q}$.

The inductive step $\exists$. Assuming that the assertion is valid for a formula $\Phi(f)$ for an arbitrary $f \in {}^*M$, we will prove it for the formula $\exists x \Phi(x)$. Denote $r = r(\Phi)$.

From left to right. Suppose that $\exists x\ \Phi(x)$ is true in ${}^*M$; i.e., $\Phi(f)$ holds with a suitable $f \in {}^*M$. Let $p = \max\{r, r(f)\}$. To diminish bulkiness, we conventionally agree to denote by $\mathbf{i}$ and $\mathbf{j}$ finite sequences of the forms

$$\langle i_1, \ldots, i_r \rangle \in I^r \text{ and } \langle i_1, \ldots, i_r, \ldots, i_p \rangle \in I^p,$$

respectively, and to think of the expressions $\mathbf{Qi}$ and $\mathbf{Qj}$ as the sequences

$$\mathbf{Q}i_r \ldots \mathbf{Q}i_1 \text{ and } \mathbf{Q}i_p \ldots \mathbf{Q}i_r \ldots \mathbf{Q}i_1.$$

By the induction hypothesis, we have $\mathbf{Qj}\ \Phi(f)[\mathbf{j}]$. Besides, it is clear that $\Phi(f)[\mathbf{j}] \to \exists x\ \Phi(x)[\mathbf{j}]$ for all $\mathbf{j}$. Consequently, $\mathbf{Qj}\ \exists x\ \Phi(x)[\mathbf{j}]$. However, we have $r(\exists x\ \Phi(x)) = r \leq p$; therefore, the formula $\exists x\ \Phi(x)[\mathbf{j}]$ coincides simply with $\exists x\ \Phi(x)[\mathbf{i}]$. Thus, the superfluous quantifiers $\mathbf{Q}$ can be eliminated, on replacing the prefix $\mathbf{Qj}$ by $\mathbf{Qi}$.

From right to left. Suppose that $\mathbf{Qi}\ \exists x\ \Phi(x)[\mathbf{i}]$ holds. The set

$$P = \{ \langle \mathbf{i}, x \rangle : \mathbf{i} \in I^r \& x \in M \text{ and } \Phi(x)[\mathbf{i}] \text{ is true in } M \}$$

belongs to Def(V) by (Q2). For each $\mathbf{i} \in I^r$, we denote by $f(\mathbf{i})$ the set $p \in M$ that is the least element (with respect to the canonical well ordering mentioned in Subsection 3.1 and given by the axiom of constructibility) among those satisfying $\langle \mathbf{i}, p \rangle \in P$, in case such a set p exists; while $f(\mathbf{i}) = \varnothing$, otherwise. Taking into account the definability in V of the indicated well ordering restricted to V, we conclude that f is definable in V; i.e., $f \in {}^*M$. However, by definition,

$$\forall \mathbf{i} \in I^r\ (\exists x \Phi(x)[\mathrm{i}] \to \Phi(f)[\mathrm{i}]);$$

whence $\mathbf{Qi}\ \exists x\ \Phi(x)[\mathrm{i}] \to \mathbf{Qi}\ \Phi(f)[\mathrm{i}]$. The left-hand side of the last implication coincides with the right-hand side of the equivalence of the theorem for the formula $\exists x\ \Phi(x)$ and, consequently, the left-hand side is true by our assumption. Therefore, the right-hand side is true as well. Hence, by the induction hypothesis, we have $\Phi(f)$ in ${}^*M$ and, finally, $\exists x\ \Phi(x)$ in ${}^*M$. □

Corollary. *Let φ be a closed $\in$-formula with parameters in M and let ${}^*\varphi$ be obtained from φ by replacing each parameter $p \in M$ with ${}^*p$. Then*

φ holds in M if and only if ${}^*\varphi$ holds in ${}^*M$.

Proof. It is clear, for ${}^*\varphi[\mathbf{i}]$ coincides with φ. □

Theorem 9. *$\langle {}^*M, {}^*{=}, {}^*{\in}, {}^*\mathrm{st} \rangle$ is a model of* IST.

Proof. The corollary to the Łoš theorem means nothing else than the validity of the transfer principle in ${}^*M$. Hence all the axioms of ZFC are also valid in ${}^*M$. The standardization principle in ${}^*M$ is guaranteed by the following property of the initial model: if $y \subseteq x \in M$ then $y \in M$. So we are left with proving only the idealization principle.

Thus, let $\varphi(x, a)$ be a $\in$-formula with parameters in ${}^*M$ and let $r = r(\varphi)$. We are to prove that

$$\forall^{\mathrm{st\,fin}} A\ \exists x\ \forall a \in A\ \varphi(x, a) \to \exists x\ \forall^{\mathrm{st}} a\ \varphi(x, a)$$

holds in ${}^*M$ (the reverse implication in (I) needs no separate examining; for, it ensues easily from the standardization principle which implies that all the elements of any standard set are standard [1]).

In accord with the Łoš theorem, the left-hand side of the implication can be rewritten in the form

$$\forall^{\mathrm{fin}} A \subseteq M\ \mathbf{Q}i_r\ \ldots\ \mathbf{Q}i_1\ \exists x\ \forall a \in A\ (\varphi(x, a)[i_1, \ldots, i_r]).$$

Recall that I consists of all the finite subsets of M; therefore, it is quite possible for us to replace the letter A with i, bearing in mind that $i \in I$. Define a function $\tilde{A} : I^{r+1} \to M$ by setting $\tilde{A}(i_1, \ldots, i_r, i) = i$. Then the left-hand side of (I) takes the form

$$\forall i \mathbf{Q}i_r\ \ldots\ \mathbf{Q}i_1\ (\exists x\ \forall a \in \tilde{A}\ \varphi(x, a))[i_1, \ldots, i_r, i].$$

Certainly, we can replace $\forall i$ with $\mathbf{Q}i$. Again from the Łoš theorem it follows that $\exists x\ \forall a \in \tilde{A}\ \varphi(x, a) \in {}^*M$.

Now, in view of the definition of standardness in ${}^*M$, for the derivation of the right-hand side of (I), it suffices to check that ${}^*a \in \tilde{A}$ is true in ${}^*M$ for an arbitrary $a \in M$. By the Łoš theorem, this is tantamount to

$$\mathbf{Q}i\ \mathbf{Q}i_r\ \ldots\ \mathbf{Q}i_1\ (a \in \tilde{A}[i_1, \ldots, i_r, i]),$$

i.e. $\mathbf{Q}i\ \mathbf{Q}i_r\ \ldots\ \mathbf{Q}i_1\ (a \in i)$ by the definition of $\tilde{A}$; and the last ensues immediately from property (Q1). □

3.5. Violation of (E1) in ${}^*M$. Returning to the cardinals $\varkappa_n$ of Subsection 3.1, we see that, for each integer n, there is a certain $\in$- formula $\Phi_n(\varkappa)$ which defines the cardinal $\varkappa_n$ in the sense that the following assertion is true: $\varkappa_n$ is the unique set for which $\Phi_n(\varkappa)$ holds. This claim together with Lemma 6 will play the central role in our proving that (E1) fails in ${}^*M$.

We shall actually obtain the falsity of (E1) in two different forms: first, with the uniqueness requirement on the left-hand side, i.e., in the form (E1!), but for a formula Φ which is more involved than formulas of type Π_2^{st}; second, for a matrix formula of type Π_2^{st} exactly but without uniqueness.

Thus, let $\tau(\cdot)$ be the formula given by Theorem 4 of §2. Denote by $\Phi(n,\varkappa)$ the formula

$$n \in \mathbb{N} \ \& \ \mathrm{st}\, n \ \& \ \tau(\ulcorner \Phi_n(\varkappa) \urcorner) \ \& \ \mathrm{st}\, \varkappa.$$

Theorem 10. *The following example of* (E1!) *fails in* $^*M$:

$$\forall^{\mathrm{st}} n \in \mathbb{N}\ \exists! \varkappa\ \Phi(n,\varkappa) \rightarrow \exists \tilde{\varkappa}\ \forall^{\mathrm{st}} n \in \mathbb{N}\, \Phi(n, \tilde{\varkappa}(n)).$$

Proof. Validate the left-hand side in $^*M$: To this end, given a fixed integer n, prove that $\exists! \varkappa \Phi(^*n, \varkappa)$ in $^*M$. To verify existence for $\varkappa$, we take $\varkappa = {}^*\varkappa_n$. Then $\Phi_n(\varkappa)$ holds in M and, consequently, $\Phi_n(^*\varkappa)$ holds in $^*M$ by transfer. From the choice of the formula τ and the definition of Φ it ensues that $\Phi(^*n, {}^*\varkappa)$ holds in $^*M$, as required. To prove uniqueness, suppose that $\Phi(^*n, \varkappa')$ is true in $^*M$. Then $\varkappa'$ is a standard cardinal in $^*M$; i.e., we can assume that $\varkappa' = {}^*\varkappa$ for some cardinal $\varkappa \in M$. Reversing the foregoing arguments, we deduce that $\varkappa = \varkappa_n$.

Invalidate the right-hand side: Suppose on the contrary that $\tilde{\varkappa} \in {}^*M$ is such that $\Phi(^*n, \tilde{\varkappa}(^*n))$ holds in $^*M$ for any n. Put $r = r(\tilde{\varkappa})$. By the Łoš theorem, we have

$$\varkappa = \varkappa_n \leftrightarrow \mathbf{Q} i_r \ \ldots \ \mathbf{Q} i_1\ (^*\varkappa = \tilde{\varkappa}(^*n))[i_1, \ldots, i_r]$$

in V for any n and $\varkappa$. However, $\tilde{\varkappa} \in \mathrm{Def}(V)$, the map $s \mapsto {}^*s$ belongs to $\mathrm{Def}(V)$, and $\mathrm{Def}(V)$ is closed under the action of the quantifier $\mathbf{Q}$ by the property (Q2) of $\mathbf{Q}$. Hence, the sequence $\langle \varkappa_n : n \in \mathbb{N} \rangle$ belongs to $\mathrm{Def}(V)$, which contradicts Lemma 6. □

To construct the second example of refutation for (E1), we use a slightly different matrix formula; namely,

$$\varphi(n,T) =^{\mathrm{def}} n \in \mathbb{N} \ \& \ \mathrm{st}\, n \ \& \ \mathrm{Sat}(T) \ \& \ \exists^{\mathrm{st}} \varkappa\ (\ulcorner \Phi_n(\varkappa) \urcorner \in T).$$

Theorem 11. *The following example of* (E1) *fails in* $^*M$:

$$\forall^{\mathrm{st}} n \in \mathbb{N}\ \exists T\ \varphi(n,T) \rightarrow \exists \tilde{T}\ \forall^{\mathrm{st}} n \in \mathbb{N}\ \varphi(n, \tilde{T}(n)).$$

Proof. Validate the left-hand side: Given a natural n, the set $\tilde{T}$ fitting the left-hand side appears by applying (in the framework of $^*M$) Claim 1 of the proof of Lemma 5 to the formula $\Phi_n(^*\varkappa_n)$.

Invalidate the right-hand side: Were there an element $\tilde{T} \in {}^*M$ possessing the indicated property, we would again obtain the definability in V of the sequence of cardinals $\varkappa_n$; for, we have

$$\varkappa = \varkappa_n \leftrightarrow \mathbf{Q} i_r \ \ldots \ \mathbf{Q} i_1\ (\ulcorner \Phi_n(^*\varkappa) \urcorner \in \tilde{T}(^*n))[i_1, \ldots, i_r]$$

for any n and $\varkappa$ (on applying further the property (Q2) of the quantifier $\mathbf{Q}$, which yields a contradiction to Lemma 6). □

Now, a few words about complexity of the matrix formulas Φ and φ used in our counterexamples. Simple analysis shows that the formula Sat of §2 is of type Π_2^{st} (to be more precise, it is expressible as a formula of type Π_2^{st} with the help of elementary transformations). It follows that φ is also a formula of type Π_2^{st} (in the same sense). Furthermore, the truth formula τ of §2 is constructed from Sat in such a way that it can be rewritten in either of the following two forms: $\exists T\ \sigma(T,\cdot)$ and $\forall T\ \pi(T,\cdot)$, where σ and π are formulas of types Π_2^{st} and Σ_2^{st}, respectively. It is convenient to denote these two types of formulas by $\exists\Pi_2^{\mathrm{st}}$ and $\forall\Sigma_2^{\mathrm{st}}$. Thus, the formula Φ of Subsection 3.5 can be transformed to either of the two forms: $\exists\Pi_2^{\mathrm{st}}$ and $\forall\Sigma_2^{\mathrm{st}}$.

A natural question arises whether it is possible to refute (E1!) (i.e., with uniqueness on the left-hand side) for a formula Φ of type Π_2^{st}. The answer is negative as least for the formulas Φ having only standard parameters. The author succeeded in proving (E1!) in IST for all formulas Φ, with only standard parameters, of the form $\mathrm{Q}\chi$, where χ is an internal formula and Q is a quantifier prefix containing only external quantifiers (in the sense indicated in the Introduction). Theorem 11 also cannot be improved in the case of standard parameters; for, it happens that (E1) holds in IST for all matrix Σ_2^{st} formulas with standard parameters.

3.6. Violation of (E) in $^*M$. The refutation of (E) will be also given in two variants: first, assuming uniqueness on the left-hand side; second, in the general case. We will make use of a finite set H such that $\mathbb{S} \subseteq H$ (i.e., H contains all the standard sets; the existence of such a set H was proved in [1]). Let ν denote the number of elements in H, let $K = \{1, 2, \dots, \nu\}$, and let h be a bijection of K onto H. Consider the formula

$$\Psi(n, k) =^{\mathrm{def}} k \in K \;\;\&\;\; \Phi(n, h(k)) \;\;\&\;\; \mathrm{st}\, h(k),$$

where Φ is the formula of Theorem 10, and let $X = Y = \mathbb{N}$.

Theorem 12. *The following example of* (E!) *fails in* $^*M$:

$$\forall^{\mathrm{st}} n \exists! k \;\Psi(n, k) \to \exists \tilde{k} \;\forall^{\mathrm{st}} n \;\Psi(n, \tilde{k}(n)) \quad (n, k \in \mathbb{N}).$$

Proof. Assume on the contrary that this implication holds in $^*M$. The left-hand side of (E!) is true in $^*M$ (given n fixed, take $k = h^{-1}(\varkappa_n)$). Therefore, the right-hand side must be also true; i.e., we have a function $\tilde{k} : \mathbb{N} \to K$ such that $h(\tilde{k}(n))$ is standard and satisfies $\Phi(n, h(\tilde{k}(n)))$ for all $n \in \mathbb{N}$. It suffices to put $\tilde{\varkappa}(n) = h(\tilde{k}(n))$ for all $n \in \mathbb{N}$. The function $\tilde{\varkappa}$ ensures the right-hand side of (E1!) for the formula Φ of Subsection 3.5, which contradicts Theorem 10. □

It is easily seen that the formula Ψ, together with Φ, can be related to any of the classes $\exists\Pi_2^{\mathrm{st}}$ and $\forall\Sigma_2^{\mathrm{st}}$.

To refute (E) (without uniqueness on the left-hand side) in $^*M$ for a matrix formula of type Π_2^{st}, we consider the formula

$$\psi(n, t) =^{\mathrm{def}} t \subseteq K \;\;\&\;\; \varphi(n, h''t) \qquad (\text{where } h''t = \{ h(k) : k \in t \}),$$

and the sets $X = \mathbb{N}$ and $Y = \mathcal{P}^{\mathrm{fin}}(\mathbb{N})$.

Theorem 13. *The following example of* (E) *fails in* $^*M$:

$$\forall^{\mathrm{st}} n \in \mathbb{N} \;\exists t \;\psi(n, t) \to \exists \tilde{t} \;\forall^{\mathrm{st}} n \in \mathbb{N} \;\psi(n, \tilde{t}(n)).$$

Proof. Reduce the claim to Theorem 11. □

It is clear that in the proposed method of refuting (E) in the model $^*M$ we cannot hope to eliminate the parameters ν, K, and h (those are nonstandard; moreover, h is not even an element of a standard set). It is likely that a proof of independence for (E) needs another construction for a matrix formula and, perhaps, another construction of the model itself.

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