Siberian Mathematical Journal, Vol. 58, *No.* 6, *pp.* 1012–1014, 2017
Original Russian Text Copyright © 2017 *Kanovei V.G. and Lyubetsky V.A.*

A GENERIC PROPERTY OF THE SOLOVAY SET Σ

© **V. G. Kanovei and V. A. Lyubetsky** UDC 510.223:510.225

Abstract: We prove that the Solovay set Σ is generic over the ground model in the sense of a forcing whose order relation extends the order relation of the given forcing.

DOI: 10.1134/S0037446617060106

Keywords: genericity, Solovay set Σ

1. Introduction. Solovay's article [1] belongs to the classics of forcing. The main result of [1] is the construction of the model indicated in its title as well as a close model in which only definable sets are assumed Lebesgue measurable but the axiom of choice is true, in contrast to the first model, in which it is necessarily violated. In [1], Solovay introduced several methods in set theory that are key for the subsequent development of the forcing method. One of them was the "important lemma" from [1, Section 4.4], which states that any generic extension of the ground model is also a generic extension of any intermediate model in the sense of a certain forcing Σ that is a subset of the given forcing.

In Theorem 3 of this article, we prove that, in the context of Solovay's construction, the set Σ is itself generic over the ground model in the sense of a forcing $\mathbb{Q}$ closely related to the given forcing $\mathbb{P}$ in the sense that its domain is equal to the domain of the given forcing $\mathbb{P}$ but the order relation $\leq_{\mathbb{Q}}$ itself $\subseteq$-extends the order $\leq_{\mathbb{P}}$ of $\mathbb{P}$. As an application of this theorem, we present a brief proof of the following result (Theorem 4): Any submodel of a Cohen generic extension of a model is itself an extension of the ground model (or coincides with the last one), and, on the other hand, the extension itself is a Cohen generic extension of this intermediate submodel (or again coincides with the submodel). This result is known in principle to experts in forcing but we unaware of any published proof.

2. The Solovay set Σ. The above result by Solovay is as follows:

Proposition 1. *Suppose that $\mathbb{P}$ is a forcing in a set-theoretic universe $\boldsymbol{V}$, a set $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic set over $\boldsymbol{V}$, $t \in \boldsymbol{V}$ is a $\mathbb{P}$-name, and $X = t[G] \subseteq \boldsymbol{V}$ ($t[G]$ stands for the G-estimate, or the G-interpretation, of the name t; $t[G] \in \boldsymbol{V}[G]$). Then there exists a set $\Sigma = \Sigma(X,t) \in \boldsymbol{V}[G]$, $\Sigma \subseteq \mathbb{P}$, such that*

(I) *Σ is closed downward in $\mathbb{P}$, i.e., if $q \in \Sigma$, $p \in \mathbb{P}$, and $p \leq q$, then $q \in \Sigma$ ($p \leq q$ means that the forcing condition q is stronger than p);*

(II) $\boldsymbol{V}[\Sigma] = \boldsymbol{V}[X]$;

(III) *$G \subseteq \Sigma$ and G is Σ-generic over $\boldsymbol{V}[X]$;*

(IV) *the class $\boldsymbol{V}[G]$ is a Σ-generic extension of the model $\boldsymbol{V}[X] = \boldsymbol{V}[\Sigma]$;*

(V) *if a set $G' \subseteq \Sigma$ is Σ-generic over $\boldsymbol{V}[X]$ then G' is $\mathbb{P}$-generic over $\boldsymbol{V}$ and, moreover, $t[G'] = X$.*

Proof (a sketch of the construction of Σ, see [1, 4.4] or [2, 13.3.2] for a detailed proof). Put $\Sigma = \mathbb{P} \setminus \bigcup_{\xi<\vartheta} A_\xi$, where the ordinal ϑ is defined in constructing (see below) and define a sequence of sets $A_\xi \subseteq \mathbb{P}$ in $\boldsymbol{V}[G]$ as follows:

(1) A_0 consists of all "conditions" $p \in \mathbb{P}$ forcing $\check{x} \in t$ for some $x \in \boldsymbol{V} \setminus X$ or $\check{x} \notin t$ for some $x \in X$ (if $x \in \boldsymbol{V}$ then $\check{x}$ is a canonical name for x).

(2) $A_{\xi+1}$ consists of all "conditions" $p \in \mathbb{P}$ for which there exists a dense set $D \in \boldsymbol{V}$ in $\mathbb{P}$ satisfying the following: if $q \in D$ and $p \leq q$ then $q \in A_\xi$.

(3) $A_\lambda = \bigcup_{\xi<\lambda} A_\xi$ for limit ordinals λ.

V. G. Kanovei was supported by the Russian Foundation for Basic Research (Grants 13–01–00006 and 17–01–00705). V. A. Lyubetsky was supported by the Russian Science Foundation (Grant 14–50–00150).

Moscow. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 58, No. 6, pp. 1302–1305, November–December, 2017; DOI: 10.17377/smzh.2017.58.610. Original article submitted November 2, 2016.

 0037-4466/17/5806–1012

Thus, each "condition" $p \in A_0$ directly contradicts the assumption that t is a name for X by (1), and this contradiction is preserved in items (2) and (3) of the inductive definition for all large indices ξ respectively in a less and less direct form. This $\subseteq$-increasing sequence of sets $A_\xi \subseteq \mathbb{P}$ stabilizes at some limit ordinal $\vartheta \in \boldsymbol{V}$, and we have the sets $A = \bigcup_{\xi<\vartheta} A_\xi$ and $\Sigma = \mathbb{P} \setminus A$. $\square$

Subsequent works (see [3] etc.) about intermediate submodels of generic extensions showed that not only $\boldsymbol{V}[G]$ is a generic extension of an intermediate model $\boldsymbol{V}[X]$ in accordance with (III) but the intermediate model $\boldsymbol{V}[X]$ is itself a generic extension of the ground model $\boldsymbol{V}$ (see also [4, Fact 11; 5, 15.43; 6, 10.10; 7; 8, Section 1] among other references). Theorem 3 below (which is our main result) shows that in fact already $\Sigma(X,t)$ is itself a generic set over $\boldsymbol{V}$ in the sense of some forcing closely related to the given forcing $\mathbb{P}$.

3. Genericity of $\boldsymbol{\Sigma}$. Arguing in the context of Proposition 1, define a new partial order $\le_t$ on $\mathbb{P}$ that extends the given order relation $\le\, = \,\le_{\mathbb{P}}$ as follows: $p \le_t q$ if the "condition" q $\mathbb{P}$-forces over $\boldsymbol{V}$ that $\breve{p} \in \Sigma(\breve{t}[\underline{G}], \breve{t})$. In other words, for $p \le_t q$ it is necessary and sufficient that the relation $p \in \Sigma(X,t)$ be fulfilled whenever the set $G \subseteq \mathbb{P}$ is generic over $\boldsymbol{V}$, $X = t[G]$, and $q \in G$.

Lemma 2. *The relation $\le_t$ is a partial order relation on $\mathbb{P}$, belongs to $\boldsymbol{V}$, and extends the given order $\le\, = \,\le_{\mathbb{P}}$, i.e., $\le_{\mathbb{P}} \subseteq \le_t$ (or, equivalently, $p \le_{\mathbb{P}} q$ implies $p \le_t q$).*

Proof. Suppose that $p \le_t q \le_t r$. For deducing $p \le_t r$, assume that a set $G \subseteq \mathbb{P}$ is generic over $\boldsymbol{V}$, $r \in G$, and $X = t[G]$; it is required to check that $p \in \Sigma(X,t)$.

By definition, $q \in \Sigma(X,t)$. Consider any set $G' \subseteq \Sigma(X,t)$ $\Sigma(X,t)$-generic on $\boldsymbol{V}[X]$ and containing q. Then G' is $\mathbb{P}$-generic over $\boldsymbol{V}$ and $t[G'] = X$ by (V). Hence, $p \in \Sigma(X,t)$, since $p \le_t q$, which was required.

Finally, assume that $p \le q$ and deduce $p \le_t q$. Consider any set $G \subseteq \mathbb{P}$ generic over $\boldsymbol{V}$ and containing $q \in G$ and let $X = t[G]$. For proving $p \in \Sigma(X,t)$, observe that $q \in \Sigma(X,t)$ by (III); now, $p \in \Sigma(X,t)$ by (I). $\square$

Theorem 3 (under the conditions of Proposition 1). *Suppose that $G \subseteq \mathbb{P}$ is a generic set over $\boldsymbol{V}$ and $X = t[G]$. The the set $\Sigma = \Sigma(X,t)$ is itself generic over $\boldsymbol{V}$ in the sense of the forcing $\mathbb{P}_t = \langle \mathbb{P}; \le_t \rangle$.*

Proof. Assume that $p \in \mathbb{P}$, $q \in \Sigma$, and $p \le_t q$. For deducing $p \in \Sigma$, consider any set $G' \subseteq \Sigma$ that is Σ-generic over $\boldsymbol{V}[X]$ and contains q. Then G' is $\mathbb{P}$-generic over $\boldsymbol{V}$ and $t[G'] = X$ by (V). Hence, $p \in \Sigma$ because $p \le_t q$.

Prove that every two "conditions" $p, q \in \Sigma$ are $\le_t$-compatible in Σ. By genericity, there is a "condition" $r \in G$ forcing both $\breve{p} \in \Sigma(\breve{t}[\underline{G}], \breve{t})$ and $\breve{q} \in \Sigma(\breve{t}[\underline{G}], \breve{t})$. Then, by definition, $p \le_t r$ and $q \le_t r$, and, on the other hand, $r \in \Sigma$ by (III).

Check the genericity condition. Consider an arbitrary $\le_t$-dense set $D \subseteq \mathbb{P}$ (not necessarily dense with respect to the original order $\le$). Suppose on the contrary that some "condition" $p \in G$ forces that $\breve{D} \cap \Sigma(\breve{t}[\underline{G}], \breve{t}) = \varnothing$. By density, there exists a "condition" $q \in D$ satisfying $p \le_t q$.

Now, consider an arbitrary set $G' \subseteq \mathbb{P}$ $\mathbb{P}$-generic over $\boldsymbol{V}$ and containing q and let $X' = t[G']$. Then $q \in G' \subseteq \Sigma' = \Sigma(X', t)$, whence, by what was proved, $p \in \Sigma'$.

Finally, consider a set $G'' \subseteq \Sigma'$ Σ'-generic over $\boldsymbol{V}[X']$ and containing p. Then G'' is $\mathbb{P}$-generic over $\boldsymbol{V}$ and $t[G''] = X'$ over (V). Thus, $q \in D \cap \Sigma(t[G''], t)$ and $p \in G''$, and we get a contradiction to the choice of the "condition" p. $\square$

4. Intermediate submodels of Cohen generic extensions. Recall that Cohen generic extensions use the forcing $\mathbb{C} = 2^{<\omega}$ (all finite dyadic sequences).

Theorem 4 (a forcing folklore). *Suppose that a point $a \in 2^\omega$ is Cohen generic over a universe $\boldsymbol{V}$ and $X \in \boldsymbol{V}[a]$, $X \subseteq \boldsymbol{V}$. Then*

(i) *either $X \in \boldsymbol{V}$ or the model $\boldsymbol{V}[X]$ is a Cohen generic extension of the universe $\boldsymbol{V}$;*

(ii) *either $\boldsymbol{V}[X] = \boldsymbol{V}[a]$ or $\boldsymbol{V}[a]$ is a Cohen generic extension of the model $\boldsymbol{V}[X]$.*

Proof. The Cohen forcing $\mathbb{C} = 2^{<\omega}$ is obviously countable. Therefore, for $\mathbb{P} = \mathbb{C}$ both the forcing Σ in Proposition 1 and the forcing $\mathbb{Q}$ in Theorem 3 are countable and, as is known, a countable forcing is either trivial or produces a Cohen extension. $\square$

References

1. Solovay R. M., "A model of set theory in which every set of reals is Lebesgue measurable," Ann. Math., vol. 92, 1–56 (1970).
2. Kanoveĭ V. G. and Lyubetskiĭ V. A., *Modern Set Theory: Absolutely Unsolvable Classical Problems* [Russian], MTsNMO, Moscow (2013).
3. Grigorieff S., "Intermediate submodels and generic extensions of set theory," Ann. Math., vol. 101, 447–490 (1975).
4. Fuchs G., Hamkins J. D., and Reitz J., "Set-theoretic geology," Ann. Pure Appl. Logic, vol. 166, no. 4, 464–501 (2015).
5. Jech T., *Set Theory. The Third Millennium Revised and Expanded*, Springer-Verlag, Berlin, Heidelberg, and New York (2003).
6. Kanamori A., *The Higher Infinite. Large Cardinals in Set Theory from Their Beginnings*, Springer, Berlin (2003).
7. Kanoveĭ V. G. and Lyubetsky V. A., "Generalization of one construction by Solovay," Sib. Math. J., vol. 56, no. 6, 1072–1079 (2015).
8. Zapletal J., "Terminal notions in set theory," Ann. Pure Appl. Logic, vol. 109, no. 1–2, 89–116 (2001).

V. G. Kanovei; V. A. Lyubetsky
Institute for Information Transmission Problems
Russian University of Transport, Moscow, Russia
E-mail address: `kanovei@mccme.ru; lyubetsk@ippi.ru`