ISSN 0037–4466, Siberian Mathematical Journal, 2019, Vol. 60, No. 6, pp. 1003–1006. © Pleiades Publishing, Ltd., 2019.
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ABSOLUTENESS OF THE SOLOVAY SET Σ

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UDC 510.223; 510.225

Abstract: We prove that the Solovay set Σ is absolutely definable in a sufficiently wide sense; in particular, Σ does not depend on the choice of the ground model.

DOI: 10.1134/S0037446619060089

Keywords: absoluteness, genericity, Solovay set Σ

1. Introduction

The main result of Solovay's classical article [1] consists in constructing a model as given in its title and a second model in which only definable sets are measurable but the axiom of choice holds in contrast to the first model where it fails. The key role in the construction of these models was played by the "Important Lemma" of [1, Section 4.4], by which each generic extension of the ground model is a generic extension of any intermediate model.

More exactly, if $\mathbb{P} = \langle \mathbb{P}; \leqslant \rangle$[1] is a forcing in a set-theoretic universe $\mathbf{V}$, while $t \in \mathbf{V}$ is a simple $\mathbb{P}$-name (i.e., $t \subseteq \mathbb{P} \times \mathbf{V}$), $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathbf{V}$, and $X = t[G]$ is the corresponding *valuation*,[2] then there is $\Sigma = \Sigma^{\mathbf{V}}_{\mathbb{P}t}(X) \subseteq \mathbb{P}$ for which $\mathbf{V}[\Sigma] = \mathbf{V}[X]$, $G \subseteq \Sigma$, and G is Σ-generic over $\mathbf{V}[X]$. Thus, $\mathbf{V}[G]$ is a Σ-generic extension of the intermediate model $\mathbf{V}[X] \subseteq \mathbf{V}[G]$.

The structure of intermediate models was considered later in [2–6] and elsewhere as well as in [7], where it was established that, under the above conditions, $\Sigma = \Sigma^{\mathbf{V}}_{\mathbb{P}t}(X)$ is generic over $\mathbf{V}$ in the sense of the forcing $\mathbb{P}_t = \langle \mathbb{P}; \leqslant_t \rangle$ with the same domain $\mathbb{P}$ and order relation $\leqslant_t = \leq^{\mathbf{V}}_{\mathbb{P}t}$, which $\subseteq$-extends the given order $\leqslant$. Thus, the intermediate model $\mathbf{V}[X] = \mathbf{V}[\Sigma]$ becomes a generic extension of a given universe $\mathbf{V}$.

The investigations of recent years [8, 9] showed that, while dealing with complex iterated generic extensions, the important role is played by the *absoluteness* of the definition of $\Sigma^{\mathbf{V}}_{\mathbb{P}t}(X) \subseteq \mathbb{P}$ and the order $\leq^{\mathbf{V}}_{\mathbb{P}t}$ in the sense of independence of $\mathbf{V}$. (Dependence on $\mathbb{P}$, t, and X is obvious and unremovable.) This independence was established in [9] *ad hoc* in one particular case. Here we prove the general result:

Theorem 1. *Suppose that $\mathbb{P} = \langle \mathbb{P}; \leqslant \rangle \in \mathbf{V}$ is a forcing in a universe $\mathbf{V}$, while $t \in \mathbf{V}$ is a $\mathbb{P}$-name, $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathbf{V}$, and $X = t[G] \subseteq \mathbf{V}$. Assume in addition that $\mathbf{V}$ is a generic extension of the class $\mathbf{L}[\mathbb{P}, t] \subseteq \mathbf{V}$. Then $\Sigma^{\mathbf{V}}_{\mathbb{P}t}(X) = \Sigma^{\mathbf{L}[\mathbb{P},t]}_{\mathbb{P}t}(X)$ and the relation $\leq^{\mathbf{V}}_{\mathbb{P}t}$ is identical to $\leq^{\mathbf{L}[\mathbb{P},t]}_{\mathbb{P}t}$.*

This theorem finds applications in the study of the intermediate models of extensions generic in the sense of the Solovay "random" forcing (see [10]), which is one of the most applicable forcings in modern set theory[3] and which can find applications to models of other generic extensions such, for example, as the generic extensions considered in [13, 14].

2. The Solovay Set Σ and the Relation $\leqslant_t$

The definition of $\Sigma = \Sigma^{\mathbf{V}}_{\mathbb{P}t}(X)$ in the sense of [1] is as follows:

V. G. Kanovei was supported by the Russian Foundation for Basic Research (Grant 17–01–00705). V. A. Lyubetsky was supported by the Russian Foundation for Basic Research (Grant 18–29–13037).

[1] $p \leqslant q$ means that the forcing condition q is stronger than p.

[2] $t[G] = \{x : \exists p \in G\, (\langle p, x \rangle \in t)\} \in \mathbf{V}[G]$ is a *G-valuation* for t; $t[G] \subseteq \mathbf{V}$.

[3] See, for example, our article [11] on some applications of the Solovay "random" forcing. Avoiding further references, we just mention the unpublished study by E. I. Gordon [12] where random forcing is used for proving several theorems about extensions of Haar measures.

Original article submitted December 19, 2018; revised April 10, 2019; accepted May 15, 2019.

Definition 2 (under the conditions of Theorem 1). Put $\Sigma = \Sigma^{\mathbf{V}}_{\mathbb{P}t}(X) =_{\rm def} \mathbb{P} \setminus \bigcup_{\xi<\vartheta} A_\xi$, where the ordinal ϑ is defined in construction and the set sequence $A_\xi = A^{\mathbf{V}}_\xi \subseteq \mathbb{P}$ is defined in $\mathbf{V}[X]$ by induction:

(1) A_0 consists of all "conditions" $p \in \mathbb{P}$ forcing $\breve{x} \in t$ for some $x \in \mathbf{V} \setminus X$ or $\breve{x} \notin t$ for some $x \in X$.[4)]

(2) $A_{\xi+1}$ consists of all "conditions" $p \in \mathbb{P}$ for which there exists a dense set $D \in \mathbf{V}$ in $\mathbb{P}$ satisfying the following: if $q \in D$ and $p \le q$ then $q \in A_\xi$.

(3) $A_\lambda = \bigcup_{\xi<\lambda} A_\xi$ for limit ordinals λ.

Thus, each "condition" $p \in A_0$ directly contradicts the assumption that t is a name for X, by (1), and this contradiction is preserved by items (2) and (3) of the inductive definition for larger and larger indices ξ, respectively, in a more and more indirect form. This $\subseteq$-increasing set sequence $A_\xi \subseteq \mathbb{P}$ stabilizes at some limit ordinal $\vartheta \in \mathbf{V}$, and we have the sets $A = \bigcup_{\xi<\vartheta} A_\xi$ and $\Sigma = \mathbb{P} \setminus A$.

The relation $\leqslant_t \,=\, \le^{\mathbf{V}}_{\mathbb{P}t}$, which is the second subject of our interest, is introduced as follows:

Definition 3 ([7], under the conditions of Theorem 1). If $p, q \in \mathbb{P}$ then $p \leqslant_t q$ is defined when the "condition" q $\mathbb{P}$-forces over $\mathbf{V}$ that $\breve{p} \in \Sigma^{\breve{\mathbf{V}}}_{\breve{\mathbb{P}},\breve{t}}(\breve{t}[\underline{G}])$.

In other words, for $p \le^{\mathbf{V}}_{\mathbb{P}t} q$, it is necessary and sufficient that $p \in \Sigma^{\mathbf{V}}_{\mathbb{P}t}(X)$ holds every time when $G \subseteq \mathbb{P}$ is generic over $\mathbf{V}$, with $X = t[G]$, and $q \in G$.

The following two results express the key properties of $\Sigma^{\mathbf{V}}_{\mathbb{P}t}(X)$ and the relations $\le^{\mathbf{V}}_{\mathbb{P}t}$ in the context of forcing. They are not used in the proof of Theorem 1 and are given just for completeness of exposition.

Proposition 4 (see [1, 4.4] or [15, 13.3.2]). *In the notations of Definition 2, the set $\Sigma = \Sigma^{\mathbf{V}}_{\mathbb{P}t}(X)$ satisfies the conditions:*

(i) *Σ is closed downwards in $\mathbb{P}$; i.e., if $q \in \Sigma$, $p \in \mathbb{P}$, and $p \le q$ then $p \in \Sigma$;*

(ii) *$\mathbf{V}[\Sigma] = \mathbf{V}[X]$;*

(iii) *$G \subseteq \Sigma$ and G is Σ-generic over $\mathbf{V}[X]$; therefore, $\mathbf{V}[G]$ is a Σ-generic extension of the model $\mathbf{V}[X] = \mathbf{V}[\Sigma]$;*

(iv) *if $G' \subseteq \Sigma$ is Σ-generic over $\mathbf{V}[X]$ then G' is $\mathbb{P}$-generic over $\mathbf{V}$ too and $t[G'] = X$.* □

Proposition 5 (see [7]). (i) *$\le^{\mathbf{V}}_{\mathbb{P}t}$ is a partial order on $\mathbb{P}$. It belongs to $\mathbf{V}$ and extends the given order $\leqslant = \leqslant_{\mathbb{P}}$, i.e., $\leqslant \subseteq \le^{\mathbf{V}}_{\mathbb{P}t}$ (or, equivalently, $p \leqslant q$ that implies $p \le^{\mathbf{V}}_{\mathbb{P}t} q$).*

(ii) *If $G \subseteq \mathbb{P}$ is generic over $\mathbf{V}$ and $X = t[G]$ then $\Sigma = \Sigma^{\mathbf{V}}_{\mathbb{P}t}(X)$ is generic over $\mathbf{V}$ in the sense of the forcing $\langle \mathbb{P}; \le^{\mathbf{V}}_{\mathbb{P}t}\rangle$.* □

3. Absoluteness of Σ and $\leqslant_t$

Proof of Theorem 1. Part 1. Let us prove that $\Sigma^{\mathbf{V}}_{\mathbb{P}t}(X) = \Sigma^{\mathbf{L}[\mathbb{P},t]}_{\mathbb{P}t}(X)$. By hypothesis, there exist a forcing $Q = \langle Q; \le_Q\rangle \in \mathbf{L}[\mathbb{P}, t]$ and $H \subseteq Q$ Q-generic over $\mathbf{L}[\mathbb{P}, t]$ satisfying $\mathbf{V} = \mathbf{L}[\mathbb{P}, t][H]$. Recall that by hypothesis G is $\mathbb{P}$-generic over $\mathbf{V}$; which, by the theorem on the product of forcings, implies that H is Q-generic over $\mathbf{L}[\mathbb{P}, t][G]$ too.

Returning to Definition 2, prove by induction that $A^{\mathbf{V}}_\xi = A^{\mathbf{L}[\mathbb{P},t]}_\xi$.

It suffices to consider the induction step $\xi \to \xi + 1$ in item (2) of Definition 2. Thus, suppose that $A^{\mathbf{V}}_\xi = A^{\mathbf{L}[\mathbb{P},t]}_\xi = A_\xi$, and it is required that $A^{\mathbf{V}}_{\xi+1} = A^{\mathbf{L}[\mathbb{P},t]}_{\xi+1}$. Since $\mathbf{L}[\mathbb{P}, t] \subseteq \mathbf{V}$, we have $A^{\mathbf{L}[\mathbb{P},t]}_{\xi+1} \subseteq A^{\mathbf{V}}_{\xi+1}$. For deducing the reverse inclusion assume that $p_0 \in A^{\mathbf{V}}_{\xi+1}$ by means of a dense set $D \in \mathbf{V}$, $D \subseteq \mathbb{P}$, as in Definition 2(2). It is required to deduce $p_0 \in A^{\mathbf{L}[\mathbb{P},t]}_{\xi+1}$.

We have $D = \tau[H]$, where $\tau \in \mathbf{L}[\mathbb{P}, t]$, $\tau \subseteq Q \times \mathbb{P}$ (a suitable simple Q-name for $\mathbb{P}$). Assume without loss of generality that

$$(\langle q, p\rangle \in \tau \wedge q' \in Q \wedge q \le_Q q') \Longrightarrow \langle q', p\rangle \in \tau. \qquad (*)$$

4) If $x \in \mathbf{V}$ then $\breve{x} = \mathbb{P} \times x$ is a canonical name for x.

There is a "condition" $q_0 \in H$ Q-forcing

$$\tau[\underline{H}] \text{ is dense} \wedge \forall p \in \tau[\underline{H}](p_0 \leqslant p \Longrightarrow p \in \breve{A_\xi}) \tag{$\dagger$}$$

over $\mathbf{L}[\mathbb{P}, t[X]$, where $\breve{A_\xi} = Q \times A_\xi$ is as usual a canonical Q-name for the set $A_\xi \in \mathbf{L}[\mathbb{P}, t][X]$. Consider the sets

$$D'_1 = \{p \in \mathbb{P} : p_0 \leqslant p \wedge \exists q \in Q\,(q_0 \leq_Q q \;\wedge\; \langle q, p\rangle \in \tau)\}$$

and $D'_2 = \{p \in \mathbb{P} : p_0,\ p \text{ are incompatible in } \mathbb{P}\}$.[5)] It is clear that these sets belong to $\mathbf{L}[\mathbb{P}, t]$ as well as their union $D' = D'_1 \cup D'_2$.

We assert that D' is dense in $\mathbb{P}$. Indeed, let $p \in \mathbb{P}$. If p is incompatible with p_0 then at once $p \in D'$. If p is compatible with p_0 then we may assume that $p_0 \leqslant p$. Since q_0 forces ($\dagger$), there is a "condition" $q \in H$ such that $q_0 \leq_Q q$ and also some $p' \in \mathbb{P}$ for which $p \leqslant p'$ and q forces $p' \in \tau[\underline{H}]$. Then $\langle q, p'\rangle \in \tau$ by ($*$) and $p' \in D'$, which was required.

We state also that if $p \in D'$ and $p_0 \leqslant p$ then $p \in A_\xi$. Indeed, $p \notin D'_2$; therefore, $p \in D'_1$ by means of some "condition" $q \in Q$, and so $q_0 \leq_Q q$ and $\langle q, p\rangle \in \tau$. Then q forces $p \in \tau[\underline{H}]$; consequently, since q also forces ($\dagger$), we conclude that $p \in A_\xi$, which was required.

By what was proved, the set D' guarantees that $p_0 \in A_{\xi+1}^{\mathbf{L}[\mathbb{P},t]}$, which finishes the induction step.

Part 2. Let us prove that $\leq_{\mathbb{P}t}^{\mathbf{V}}$ is identical to $\leq_{\mathbb{P}t}^{\mathbf{L}[\mathbb{P},t]}$. Take $p, q \in \mathbb{P}$. We have to prove that the two relations are equivalent:

(A) if $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathbf{V}$, $X = t[G]$, and $q \in G$, then $p \in \Sigma_{\mathbb{P}t}^{\mathbf{V}}(X)$;

(B) if $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathbf{L}[\mathbb{P}, t]$, $X = t[G]$, and $q \in G$, then $p \in \Sigma_{\mathbb{P}t}^{\mathbf{L}[\mathbb{P},t]}(X)$.

Here (A) $\Longrightarrow$ (B) follows from the already proven equality $\Sigma_{\mathbb{P}t}^{\mathbf{V}}(X) = \Sigma_{\mathbb{P}t}^{\mathbf{L}[\mathbb{P},t]}(X)$ because every set generic over $\mathbf{V}$ is generic over the less model $\mathbf{L}[\mathbb{P}, t] \subseteq \mathbf{V}$.

For proving the reverse implication (B) $\Longrightarrow$ (A), suppose that (A) fails. This is forced by some "condition" $r \in G$, $r \geq q$, i.e., we have the following:

(C) if $G \subseteq \mathbb{P}$ is generic over $\mathbf{V}$, $X = t[G]$, and $r \in G$, then $p \notin \Sigma_{\mathbb{P}t}^{\mathbf{V}}(X)$.

Verify that then (B) fails either. To this end, consider an arbitrary set $G \subseteq \mathbb{P}$ that is $\mathbb{P}$-generic over $\mathbf{V}$ and contains r. Let $X = t[G]$. Then $p \notin \Sigma_{\mathbb{P}t}^{\mathbf{V}}(X)$ by (C); therefore, $p \notin \Sigma_{\mathbb{P}t}^{\mathbf{L}[\mathbb{P},t]}(X)$ by the above. But q also belongs to G since $q \leqslant p$. Hence, (B) fails, which was required.

4. Questions

Question 6. The premise of Theorem 1 that the given universe $\mathbf{V}$ is a generic extension of its subclass $\mathbf{L}[\mathbb{P}, t]$ is a necessary element of our proof in the part connected with D'. Does Theorem 1 hold without this premise about the genericity of $\mathbf{V}$?

Question 7. It would be interesting to obtain an analog of our absoluteness theorem in terms of the Boolean-algebraic version of forcing. Observe the following in this connection: The step from an intermediate model $\mathbf{V}[X]$ to a general extension $\mathbf{V}[G]$ is well studied in principle in the Boolean-algebraic version and amounts to replacing the Solovay set Σ by a subalgebra of the complete Boolean algebra in which the forcing $\mathbb{P}$ embeds canonically (see, for example, [16, Lemma 69]). At the same time, the step from the ground model $\mathbf{V}$ to an intermediate extension $\mathbf{V}[X]$ is not studied enough in this regard, especially, in the context of our result in [7] on the genericity of Σ in the sense of the changed order $\leqslant_t$. It is also important that complete Boolean algebras are not absolute themselves in passing from one model to another, like, for example, the set $\mathbb{R}$ of the reals is not absolute in adding a new real to the ground model. It is still unclear to us how surmountable these difficulties are on the way to a Boolean-algebraic form of our Theorem 1.

The authors are indebted to the anonymous referee for the valuable remarks that made it possible to expand and improve the exposition, in particular, to add Question 7.

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5) Incompatibility means that there is no $p' \in \mathbb{P}$ for which $p_0 \leqslant p'$ and $p \leqslant p'$.

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