A weak dichotomy below \( E_1 \times E_3 \)

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We prove that if \( E \) is an equivalence relation Borel reducible to \( E_1 \times E_3 \) then either \( E \) is Borel reducible to the equality of countable sets of reals or \( E_1 \) is Borel reducible to \( E \). The “either” case admits further strengthening.

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1. Introduction

Let \( \mathbb{R} = 2^\mathbb{N} \). Recall that \( E_1 \) and \( E_3 \) are the equivalence relations defined on the set \( \mathbb{R}^\mathbb{N} \) as follows:

\[
\begin{align*}
xE_1 y \iff & \exists k_0 \forall k > k_0 \ (x(k) = y(k)); \\
x E_3 y \iff & \forall k \ (x(k)E_0 y(k));
\end{align*}
\]

where \( E_0 \) is an equivalence relation defined on \( \mathbb{R} \) so that

\[
a E_0 b \iff \exists n_0 \forall n > n_0 \ (a(n) = b(n)).
\]

The equivalence \( E_3 \) is often denoted as \( (E_0)^\omega \).

Kechris and Louveau in [10] and Hjorth and Kechris in [3,4] proved that any Borel equivalence relation \( E \) satisfying \( E \prec_B E_1 \), resp., \( E \prec_B E_3 \), also satisfies the non-strict \( E \preceq_B E_0 \). Here \( \prec_B \) and \( \preceq_B \) are resp. strict and non-strict relations of Borel reducibility. Thus if \( E \) is an equivalence relation on a Borel set \( X^2 \) and \( F \) is an equivalence relation on a Borel set \( Y \) then \( E \preceq_B F \) means that there exists a Borel map \( \vartheta : X \to Y \) such that

\[
\begin{align*}
x \in E \Leftrightarrow & \vartheta(x) F \vartheta(x');
\end{align*}
\]

holds for all \( x, x' \in X \). Such a map \( \vartheta \) is called a (Borel) reduction of \( E \) to \( F \). If both \( E \preceq_B F \) and \( F \preceq_B E \) then they write \( E \equiv_B F \) (Borel bi-reducibility), while \( E \prec_B F \) (strict reducibility) means that \( E \preceq_B F \) but not \( F \preceq_B E \). See the cited papers [3,4] or e.g. [2,9] on various aspects of Borel reducibility in set theory and mathematics in general.

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2 We consider only Borel sets in Polish spaces.

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The above mentioned results give a complete description of the \( \leq^B \)-structure of Borel equivalence relations below \( E_1 \) and below \( E_2 \). It is then a natural step to investigate the \( \leq^B \)-structure below \( E_{13} \), where \( E_{13} = E_1 \times E_3 \) is the product of \( E_1 \) and \( E_3 \), that is, an equivalence on \( \mathbb{R}^N \times \mathbb{R}^N \) defined so that for any points \( (x, \xi) \) and \( (y, \eta) \) in \( \mathbb{R}^N \times \mathbb{R}^N \), \( (x, \xi) \sim_{E_{13}} (y, \eta) \) if and only if \( x \in E_1 y \) and \( \xi, \eta \in E_3 \).

The intended result would be that the \( \leq^B \)-cone below \( E_{13} \) includes the cones determined separately by \( E_1 \) and \( E_3 \), together with the disjoint union of \( E_1 \) and \( E_3 \) (i.e., the union of \( E_1 \) and \( E_3 \) defined on two disjoint copies of \( \mathbb{R}^N \)), \( E_{13} \) itself, and nothing else. This is however a long shot. The following theorem, the main result of this note, can be considered as a small step in this direction.

**Theorem 1.** Suppose that \( E \) is a Borel equivalence relation and \( E \leq^B E_{13} \). Then either \( E \) is Borel reducible to \( T_2 \) or \( E_1 \leq^B E \).

Recall that the equivalence relation \( T_2 \), known as "the equality of countable sets of reals", is defined on \( \mathbb{R}^N \) so that \( x T_2 y \) iff \( \{x(n): n \in \mathbb{N}\} = \{y(n): n \in \mathbb{N}\} \). It is known that \( E_3 \not\leq^B T_2 \). Theorem 1 is reduced to the following:

**Theorem 2.** Suppose that \( P_0 \subseteq \mathbb{R}^N \times \mathbb{R}^N \) is a Borel set. Then either the equivalence \( E_{13} \mid P_0 \) is Borel reducible to \( T_2 \) or \( E_1 \leq^B E_{13} \mid P_0 \).

Indeed suppose that \( Z \) (a Borel set) is the domain of \( E \), and \( \vartheta: Z \to \mathbb{R}^N \times \mathbb{R}^N \) is a Borel reduction of \( E \) to \( E_{13} \). Let \( f: Z \to 2^N = \mathbb{R} \) be an arbitrary Borel injection. Define another reduction \( \vartheta': Z \to \mathbb{R}^N \times \mathbb{R}^N \) as follows. Suppose that \( z \in Z \) and \( \vartheta(z) = (x, \xi) \in \mathbb{R}^N \times \mathbb{R}^N \). Put \( \vartheta'(z) = (x', \xi) \), where \( x' \), still a point in \( \mathbb{R}^N \), is related to \( x \) so that \( x(n) = x(n) \) for all \( n \geq 1 \) but \( x'(0) = f(z) \). Then obviously \( \vartheta(z) \) and \( \vartheta'(z) \) are \( E_{13} \)-equivalent for all \( z \in Z \), and hence \( \vartheta' \) is a Borel reduction of \( E \) to \( E_{13} \). On the other hand, \( \vartheta' \) is an injection (because so is \( f \)). It follows that its full image \( P_0 = \{z \in Z: \vartheta'(z) \in P_0 \} \) is a Borel set in \( \mathbb{R}^N \times \mathbb{R}^N \), and \( E_{13} \mid P_0 \leq^B E_{13} \mid P_0 \).

The remainder of the paper contains the proof of Theorem 2. The partition in two cases is described in Section 3. Naturally assuming that \( P_0 \) is a lightface \( \Delta^1_2 \) set, Case 1 is essentially the case when for every element \( \{x, \xi\} \in E_0 \) (note that \( x, \xi \) are points in \( \mathbb{R}^N \)) and every \( n \) we have \( x(n) = F(x, n, \xi) \mid n, \xi \mid < k \) for some \( k \), where \( F \) is a \( \Delta^1_2 \) function \( E_3 \)-invariant w.r.t. the 3rd argument. It easily follows that then the first projection of the equivalence class \( \{x, \xi\} \mid E_1 \cap P_0 \) of every point \( \{x, \xi\} \in P_0 \) is at most countable, leading to the either option of Theorem 2 in Section 5.

The results of Theorems 1 and 2 in their either parts can hardly be viewed as satisfactory because one would expect it in the form: \( E \) is Borel reducible to \( E_3 \). Thus it is a challenging problem to replace \( T_2 \) by \( E_3 \) in the theorems. Attempts to improve the either option, so far rather unsuccessful, lead to the following:

**Theorem 3.** In the either case of Theorem 2 there exist a hyperfinite equivalence relation \( G \) on a Borel set \( P''_0 \subseteq \mathbb{R}^N \times \mathbb{R}^N \) such that \( E_{13} \mid P_0 \) is Borel reducible to the least equivalence relation \( F \) on \( P''_0 \) which includes \( G \) and satisfies \( \xi \in E_3 \eta \implies \langle x, \xi \rangle \sim_F \langle y, \eta \rangle \) for all \( \langle x, \xi \rangle \) and \( \langle y, \eta \rangle \) in \( P''_0 \).

The relation \( G \) here is induced by a countable group \( G \) of homeomorphisms of \( \mathbb{R}^N \times \mathbb{R}^N \) preserving the second component. (That is, if \( g \in G \) and \( g(x, \xi) = (y, \eta) \) then \( \eta = \xi \), but \( y \) generally speaking depends on both \( x \) and \( \xi \).) And \( G \) happens to be even a locally finite group in the sense that it is equal to the union of an increasing chain of its finite subgroups. Recall that \( E_3 \) is induced by the product group \( H = (\mathcal{P}(\mathbb{N}^N), \Delta^N_1) \) naturally acting in this case on the second factor in the product \( \mathbb{R}^N \times \mathbb{R}^N \). Regarding further details see Section 6.

Case 2 is treated in Sections 7 through 12. The embedding of \( E_3 \) in \( E_{13} \mid P_0 \) is obtained by approximately the same splitting construction as the one introduced in [10] (in the version closer to [7]).

2. Preliminaries: extension of “invariant” functions

If \( E \) is an equivalence relation on a set \( X \) then, as usual, \( [x]_E = \{y \in X: x \sim y \} \) is the E-class of an element \( x \in X \), and \( [Y]_E = \bigcup_{x \in Y} [x]_E \) is the E-saturation of a set \( Y \subseteq X \). A set \( Y \subseteq X \) is E-invariant if \( Y = [Y]_E \).

The following “invariant” Separation theorem will be used below.

**Proposition 4.** (5.1 in [1]) Assume that \( E \) is a \( \Delta^1_2 \) equivalence relation on a \( \Delta^1_3 \) set \( X \subseteq \mathbb{N}^N \). If \( A, C \subseteq X \) are \( \Sigma^1_1 \) sets and \( [A]_E \cap [C]_E = \emptyset \) then there exists an E-invariant \( \Delta^1_2 \) set \( B \subseteq X \) such that \( [A]_E \subseteq B \) and \( [C]_E \cap B = \emptyset \).

Suppose that \( f \) is a map defined on a set \( Y \subseteq X \). Say that \( f \) is E-invariant if \( f(x) = f(y) \) for all \( x, y \in Y \) satisfying \( x \in E \).
Corollary 5. Assume that $E$ is a $\Delta^1_1$ equivalence relation on a $\Delta^1_1$ set $A \subseteq \mathbb{N}^\mathbb{N}$, and $f : B \to \mathbb{N}^\mathbb{N}$ is an $E$-invariant $\Sigma^1_1$ function defined on a $\Sigma^1_1$ set $B \subseteq A$. Then there exist an $E$-invariant $\Delta^1_1$ function $g : A \to \mathbb{N}^\mathbb{N}$ such that $f \leq g$.

Proof. It obviously suffices to define such a function on an $E$-invariant $\Delta^1_1$ set $Z$ such that $Y \subseteq Z \subseteq A$. (Then let $g$ be just a constant on $A \setminus Z$.) The set

$$P = \{(a, x) \in A \times \mathbb{N}^\mathbb{N} : \forall b (b \in B \land a \in E b \implies x = f(b) )\}$$

is $\Pi^1_1$ and $f \subseteq P$. Moreover $P$ is $F$-invariant, where $F$ is defined on $A \times \mathbb{N}^\mathbb{N}$ so that $\langle a, f(\alpha', y) \rangle$ if $a \in E \alpha'$ and $x = y$. Obviously $f \upharpoonright F \subseteq P$. Hence by Proposition 4 there exists an $F$-invariant $\Delta^1_1$ set $Q$ such that $f \subseteq Q \subseteq P$. Then

$$R = \{(a, x) \in Q : \forall y (y \neq x \implies \langle a, y \rangle \not\in Q )\}$$

is an $F$-invariant $\Pi^1_1$ set, and in fact a function, satisfying $f \subseteq R$. Applying Proposition 4 once again we end the proof. □

3. An important population of $\Sigma^1_1$ functions

We shall adopt a point of view where $\dim \mathbb{R}^\mathbb{N}$ is an $\Sigma^1_1$ function $\varphi : U \to \mathbb{R}$, defined on a $\Sigma^1_1$ set $U = \operatorname{dom} \varphi \subseteq \mathbb{R}^\mathbb{N} \times \mathbb{R}^\mathbb{N}$, and $\equiv_k$-invariant in the sense that if $(y, \xi)$ and $(y, \eta)$ belong to $U$ and $\xi \equiv_k \eta$ then $\varphi(y, \xi) = \varphi(y, \eta)$. Let $T \Sigma^1_n$ denote the set of all $\Delta^1_1$ functions $\psi \in T \Sigma^1_n$ with $\varphi \leq \psi$.

Lemma 7. If $\varphi \in T \Sigma^1_n$ then there is a $\Delta^1_1$ function $\psi \in T \Sigma^1_n$ with $\varphi \leq \psi$.

Proof. Apply Corollary 5. □

Definition 8. Let us fix a suitable coding system $\{W^e\}_{e \in E}$ of all $\Delta^1_1$ sets $W \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ (in particular for partial $\Delta^1_1$ functions $R \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$), where $E \subseteq \mathbb{N}$ is a $\Pi^1_1$ set, such that there exist a $\Sigma^1_1$ relation $\Sigma$ and a $\Pi^1_1$ relation $\Pi$ satisfying

$$\langle b, \xi, a \rangle \in W^e \iff \Sigma(e, b, a, \xi) \iff \Pi(e, b, a, \xi)$$

whenever $e \in E$ and $b, a \in \mathbb{R}$, $\xi \in \mathbb{R}^\mathbb{N}$.

Let us fix a $\Delta^1_1$ sequence of homeomorphisms $H_n : \mathbb{R}^{onto} \to \mathbb{R}^\mathbb{N}$. Put

$$W^e_n = \{ \langle H_n(b), \xi, a \rangle : \langle b, \xi, a \rangle \in W^e \} \quad \text{for } e \in E,$n

$$T = \{ \langle e, k \rangle : e \in E \land W^e_n \text{ is a total and } \equiv_k \text{-invariant function} \}.$$}

Here the totality means that $\operatorname{dom} W^e_n = \mathbb{R} \times \mathbb{R}^\mathbb{N}$ while the invariance means that $W^e_n(b, \xi) = W^e_n(b, \eta)$ for all $b, \xi, \eta$ satisfying $\xi \equiv_k \eta$.

Note that if $(e, k) \in T$ then, for any $n$, $W^e_n$ is a function in $T \Sigma^1_n$, and conversely, every function in $T \Sigma^1_n$ has the form $W^e_n$ for a suitable $e \in E$.

Proposition 9. $T$ is a $\Pi^1_1$ set.
Proof. Standard evaluation based on the coding of $\Delta^1_1$ sets. □

Corollary 10. The sets

$$S^k_n = \{ (x, \xi) \in \mathbb{R}_n^N \times \mathbb{R}_k^N :\exists \varphi \in \mathcal{T}_n^k \{ x(n) = \varphi(x|_{>n}, \xi) \} \}$$

$$ = \{ (x, \xi) \in \mathbb{R}_n^N \times \mathbb{R}_k^N :\exists \varphi \in \mathcal{T}_n^k \{ x(n) = \varphi(x|_{>n}, \xi) \} \}$$

belong to $\Pi^1_1$ uniformly on $n, k$. Therefore the set $S = \bigcup_m \bigcap_{n \geq m} \bigcup_k S^k_n$ also belongs to $\Pi^1_1$.

Proof. The equality of the two definitions follows from Lemma 7. The definability follows from Proposition 9 by standard evaluation.

Remark 14. Let $\varphi$ be the constant 0: $\varphi(x|_{>n}, \xi) = 0$. For any $(x, \xi) \in P_0$ put $f_{\mu}(x, \xi) = 0_{\mu(x, \xi)}(x|_{>\mu(x, \xi)})$: that is, we replace by 0 all values $x(n)$ with $n < \mu(x, \xi)$. Then $P_0' = \{ (f_{\mu}(x, \xi), (x, \xi)) : (x, \xi) \in P_0 \}$ is a $\Sigma^1_1$ set.

Put $S' = \bigcap_n \bigcup_k S^k_n$ (a $\Pi^1_1$ set by Corollary 10).

Corollary 12. There is a $\Delta^1_1$ set $P_0''$ such that $P_0'' \subseteq P_0' \subseteq S'$. The map $(x, \xi) \mapsto (f_{\mu}(x, \xi), (x, \xi))$ is a reduction of $E_{13} \upharpoonright P_0$ to $E_{13} \upharpoonright P_0''$.

Proof. Obviously $P_0''$ is a subset of the $\Pi^1_1$ set $S'$. It follows that there is a $\Delta^1_1$ set $P_0''$ such that $P_0'' \subseteq P_0'' \subseteq S'$. To prove the second claim note that $f_{\mu}(x, \xi) E_{13} x$ for all $(x, \xi) \in P_0''$. □

Let us fix a $\Delta^1_1$ set $P_0''$ as indicated. By Corollary 12 to accomplish Case 1 it suffices to get a Borel reduction of $E_{13} \upharpoonright P_0''$ to $T_2$.

Lemma 13. There exists a $\Delta^1_1$ map $\mu : P_0 \to \mathbb{N}$ such that for any $(x, \xi) \in P_0$ we have $(x, \xi) \in \bigcap_{n \geq \mu(x, \xi)} \bigcup_k S^k_n$.

Proof. Apply Kreisel Selection to the set

$$\{ (x, \xi, \eta) \in P_0 \times \mathbb{N} : \forall n \geq m \exists k (x, \xi) \in S^k_n \}.$$ □

Let $0 = \mathbb{N}^N \in \mathbb{R} = 2^\mathbb{N}$ be the constant 0: $0(k) = 0, \forall k$. For any $(x, \xi) \in P_0$ put $f_{\mu}(x, \xi) = 0_{\mu(x, \xi)}(x|_{>\mu(x, \xi)})$: that is, we replace by 0 all values $x(n)$ with $n < \mu(x, \xi)$. Then $P_0' = \{ (f_{\mu}(x, \xi), (x, \xi)) : (x, \xi) \in P_0 \}$ is a $\Sigma^1_1$ set.

Remark 14. Recall that by definition every function $F \in \mathcal{T}_n^k$ is invariant in the sense that if $(x, \xi)$ and $(x, \eta)$ belong to $\mathbb{R}_n^N \times \mathbb{R}_k^N$, $x|_{<k} = \xi|_{<k}$, and $x E_3 \eta$, then $\varphi(x, \xi) = \varphi(x, \eta)$. This allows us to sometimes use the notation $F_k(x|_{>n}, \xi|_{<k}, \xi|_{>k})$, where $k = k_j$, instead of $F_k(x|_{>n}, \xi)$, with the understanding that $F_k(x|_{>n}, \xi|_{<k}, \xi|_{>k})$ is $E_3$-invariant in the 3rd argument.

In these terms, the final equality of the lemma can be re-written as $x(n) = F_k(x|_{>n}, \xi|_{<k}, \xi|_{>k})$, where $k = k_i$.

Proof of Lemma 13. By definition $P_0'' \subseteq S'$ means that for any $(x, \xi) \in P_0''$ and $n$ there exists $k$ such that $(x, \xi) \in S^k_n$. The formula $(x, \xi) \in S^k_n$ takes the form

$$\exists \varphi \in \mathcal{T}_n^k \{ x(n) = \varphi(x|_{>n}, \xi) \}.$$ 

and further the form $\exists (e, k) \in T \{ x(n) = W^e_n(x|_{>n}, \xi) \}$. It follows that the $\Pi^1_1$ set

$$Z = \{ (x, \xi, n) \in (P_0 \times \mathbb{N}) \times T : x(n) = W^e_n(x|_{>n}, \xi) \}$$
satisfies \( \text{dom} Z = P_0 \times N \). Therefore by Kreisel Selection there is a \( \Delta_1^1 \) map \( \varepsilon : P_0 \times N \to T \) such that \( x(n) = W_n^k(x|_{>n}, \xi) \) holds for any \( (x, \xi) \in P_0 \) and \( n \), where \( (e, k) = \varepsilon(x, \xi, n) \) for some \( k \).

The range \( R = \varepsilon(\varepsilon) \) of this function is a \( \Sigma_1^1 \) subset of the \( \Pi_1^1 \) set \( T \). We conclude that there is a \( \Delta_1^1 \) set \( B \) such that \( R \subseteq B \subseteq T \). And since \( T \subseteq N \times N \), it follows, by some known theorems of effective descriptive set theory, that the set \( \tilde{E} = \text{dom} B = \{ e : \exists \, k \in B \} \) is \( \Delta_1^1 \), and in addition there exists a \( \Delta_1^1 \) map \( K : \tilde{E} \to N \) such that \( (e, K(e)) \in B \) (and \( e \in T \)) for all \( e \in \tilde{E} \).

And on the other hand it follows from the construction that
\[
\forall (x, \xi) \in P_0 \forall n \exists \varepsilon \in \tilde{E} \quad (x(n) = W_n^k(x|_{>n}, \xi)).
\]

Let us fix any \( \Delta_1^1 \) enumeration \( \{ e(i) \}_{i \in \mathbb{N}} \) of elements of \( \tilde{E} \). Put \( P_0^1 = W_{(i)}^k \). Then the last conclusion of the lemma follows from (3). Note that the functions \( P_0^1 \) are uniformly \( \Delta_1^1 \), \( P_0^1 \in \mathcal{P}_n^1 \) for some \( k \), in particular, for \( k = \kappa_i \), where \( \kappa_i = K(e(i)) \), and \( \{ \kappa_i \}_{i \in \mathbb{N}} \) is a \( \Delta_1^1 \) sequence as well. \( \square \)

**Blanket Assumption 15.** Below, we assume that the set \( P_0^\prime \) is chosen as above, that is, \( \Delta_1^1 \) and \( P_0^\prime \subseteq \mathcal{S}' \), while a system of functions \( F_0^1 \) and a sequence \( \{ \kappa_i \}_{i \in \mathbb{N}} \) of natural numbers are chosen accordingly to Lemma 13.

5. **Case 1: countability of projections of equivalence classes**

We prove here that in the assumption of Case 1 the equivalence \( E_{13} \parallel P_0^\prime \) is Borel reducible to \( T_2 \), the equality of countable sets of reals. The main ingredient of this result will be the countability of the sets
\[
C_x^\xi = \text{dom} \{ (x, \xi) \}_{E_{13}} \cap P_0^\prime = \{ y \in \mathbb{R}^N : y \in E_1 x \land \exists \eta \left( \xi \in E_3 \eta \land (y, \eta) \in P_0^\prime \right) \},
\]
where \( (x, \xi) \in P_0^\prime \) — projections of \( E_{13} \)-classes of elements of the set \( P_0^\prime \).

**Lemma 16.** If \( (x, \xi) \in P_0^\prime \) then \( C_x^\xi \subseteq [x]_{E_1} \) and \( C_x^\xi \) is at most countable.

**Proof.** That \( C_x^\xi \subseteq [x]_{E_1} \) is obvious. The proof of countability begins with several definitions. In fact we are going to organize elements of any set of the form \( C_x^\xi \) in a countable sequence.

Recall that \( \mathbb{R}^2 = 2^N \). If \( u \subseteq \mathbb{N} \) and \( b \in \mathbb{R} \) then define \( u \cdot a \) and \( (u \cdot a)(j) = a(j) \) whenever \( j \notin u \), and \( (u \cdot a)(j) = 1 - a(j) \) otherwise.

If \( f \subseteq \mathbb{N} \times \mathbb{N} \) and \( a \in \mathbb{R}^k \) then define \( f \cdot a \in \mathbb{R}^k \) so that \( (f \cdot a)(j) = (f^* a)(j) \) for all \( j < k \), where \( f^* j = \{ m : \langle j, m \rangle \in f \} \).

Note that \( f^* a \) depends in this case only on the restricted set \( f \upharpoonright k = \{ (j, m) \in f : j < k \} \).

Put \( \Phi = \mathcal{P}_{\leq n}(\mathbb{N} \times \mathbb{N}) \) and \( D = \bigcup_n D_n \), where for every \( n \):
\[
D_n = \{ (a, \varphi) : a \in \mathbb{N}^n \land \varphi \in \Phi^n \land \forall j < n (\varphi(j) \subseteq \kappa_{a(j)} \times \mathbb{N}) \}.
\]
(The inclusion \( \Phi(j) \subseteq \kappa_{a(j)} \times \mathbb{N} \) here means that the set \( \varphi(j) \subseteq \mathbb{N} \times \mathbb{N} \) satisfies \( \varphi(j) = \varphi(j) \upharpoonright \kappa_{a(j)} \), that is, every pair \( (k, l) \in \varphi(j) \) satisfies \( k < \kappa_{a(j)} \).

If \( (a, \varphi) \in D_n \), and \( (x, \xi) \in \mathbb{R}^N \times \mathbb{R}^N \) then we define \( y = r_x^\xi (a, \varphi) \in \mathbb{R}^N \) as follows: \( y = (b_0, b_1, \ldots, b_{n-1}) \upharpoonright (x|_{>n}) \), where the reals \( b_m \in \mathbb{R} \) (\( m < n \)) are defined by inverse induction so that
\[
b_m = F_m^{a(m)}((b_{m+1}, b_{m+2}, \ldots, b_{n-1}) \upharpoonright (x|_{>n}), \varphi(m), (\xi|_{<\kappa_{a(m)}}), (\xi|_{\geq \kappa_{a(m)}})).
\]
(4)

(See Remark 14 on notation. The element \( \eta = (\varphi(m) \cdot (\xi|_{<\kappa_{a(m)}}), (\xi|_{\geq \kappa_{a(m)}})) \) belongs to \( \mathbb{R}^N \) and satisfies \( \eta \in E_3 \xi \) because \( \varphi(m) \) is a finite set.)

Put \( r_x^\xi (A, \Lambda) = x \) (\( A \) is the empty sequence).

Note that by definition the function \( y \in r_x^\xi (a, \varphi) \) satisfies \( y|_{>n} = x|_{>n} \) provided \( (a, \varphi) \in D_n \), thus in any case \( x \in E_1 \) \( r_x^\xi (a, \varphi) \). Thus \( r_x^\xi \), the trace of \( (x, \xi) \), is a countable sequence, that is, a function defined on \( D = \bigcup_n D_n \), a countable set, and the set \( x \cap r_x^\xi = \{ r_x^\xi (a, \varphi) : (a, \varphi) \in D \} \) of all terms of this sequence is at most countable and satisfies \( x = r_x^\xi (\Lambda, \Lambda) \in x \cap r_x^\xi \subseteq [x]_{E_1} \).

**Claim 17.** Suppose that \( (x, \xi) \in P_0^\prime \). Then \( C_x^\xi \subseteq \text{ran} r_x^\xi \) — and hence \( C_x^\xi \) is at most countable. More exactly if \( y \in C_x^\xi \) and \( y|_{>n} = x|_{>n} \) then there is a pair \( (a, \varphi) \in D_n \) such that \( y = r_x^\xi (a, \varphi) \).

We prove the second, more exact part of the claim. By definition there is \( \eta \in \mathbb{R}^N \) such that \( (y, \eta) \in P_0^\prime \) and \( \xi \in E_3 \eta \). Put \( b_m = y(m), \forall m \). Note that for every \( m < n \) there is a number \( a(m) \) such that
Proof. The “if” direction is rather easy. If \( x \mid \eta \leq n \), then obviously \( z \in \tau x \), that is, \( z = \tau x(b, \psi) \) for a pair \( (b, \psi) \in D_m \) for some \( m \geq n \). Hence \( \tau x \subseteq \tau y \). If \( m < n \) then \( z = \tau x(b, \psi) = \tau x(a', \psi') \), where \( a' = b' \cdot (a \mid m) \) and \( \psi' = \psi \cdot (\eta \mid m) \). Then \( \tau x \subseteq \tau y \). The proof of the inverse inclusion \( \tau y \subseteq \tau x \) is similar.

Thus \( \tau x = \tau y \). It remains to prove \( \tau y \subseteq \tau x \) for all \( x, y, \eta \) such that \( \xi E_3 \eta \). Here we need another block of definitions.

Let \( H \) be the set of all \( m \in \mathbb{N} \times \mathbb{N} \) such that \( \delta = (m, j, m) \) is finite for all \( j \in \mathbb{N} \). For instance if \( \xi, \eta \in \mathbb{R}^\mathbb{N} \) satisfy \( \xi E_3 \eta \) then the set

\[
\delta_{xy} = \{ (j, m) : \xi(j) \neq \eta(j)(m) \}
\]

belongs to \( H \). The operation of symmetric difference \( \Delta \) converts \( H \) into a Polish group equal to the product group \( \langle \mathbb{Z}^\mathbb{N} \rangle : \mathbb{Z}^N \Delta \mathbb{N} \rangle \).

If \( n \in \mathbb{N} \), \( (a, \psi) \in D_n \), and \( \delta \in \mathbb{Z}^\mathbb{N} \) then we define a sequence \( \psi' = H_0^\delta(\psi) \in \mathcal{D}^n \) so that \( \psi' = (\delta \mid \xi(a)(m)) \Delta \psi(m) \) for every \( m < n \). Then the pair \( (a, H_0^\delta(\psi)) \) obviously still belongs to \( D_n \) and \( H_0^\delta(H_0^\delta(\psi)) = \psi \).

Coming back to a triple of \( y, \xi, \eta \in \mathbb{R}^\mathbb{N} \) such that \( \xi E_3 \eta \), let \( \delta = \delta_{xy} \). A routine verification shows that \( \tau y(a, \psi) = \tau y(a, H_0^\delta(\psi)) \) for all \( (a, \psi) \in D_n \). It follows that \( \tau y \subseteq \tau x \), as required. \( \square \) (Claim and Lemma 16)

The next result reduces the equivalence relation \( E_3 \upharpoonright P_0'' \) to the equality of sets of the form \( \text{ran} \tau x \), that is essentially to the equivalence relation \( T_2 \) of “equality of countable sets of reals”.

Corollary 18. Suppose that \( (x, \xi) \) and \( (y, \eta) \) belong to \( P_0'' \). Then \( (x, \xi) E_3 (y, \eta) \) holds if and only if \( \xi E_3 \eta \) and \( \text{ran} \tau x = \text{ran} \tau y \).

Proof. The “if” direction is rather easy. If \( \xi E_3 \eta \) and \( \text{ran} \tau y = \text{ran} \tau x \) then \( x \leq E_3 y \) because \( \text{ran} \tau y \subseteq \text{ran} \tau x \), and hence (see the proof of Claim 17) there exists a pair \( (a, \psi) \in D_n \) such that \( y = \tau y(a, \psi) \).

Now let us establish \( \text{ran} \tau x = \text{ran} \tau y \) (with one and the same \( \xi \)). Suppose that \( z \in \text{ran} \tau x \), that is, \( z = \tau y(b, \psi) \) for a pair \( (b, \psi) \in D_n \) for some \( m \). If \( m \neq n \) then obviously \( z = \tau y(b, \psi) = \tau x(b, \psi) \), and hence \( (\xi \mid m) = \psi \mid m \) \( z \in \text{ran} \tau x \). If \( m = n \) then \( z = \tau y(b, \psi) = \tau x(b', \psi') \), where \( a' = b' \cdot (a \mid m) \) and \( \psi' = \psi \cdot (\eta \mid m) \), and once again \( z \in \text{ran} \tau x \). Thus \( \text{ran} \tau x = \text{ran} \tau y \). The proof of the inverse inclusion \( \text{ran} \tau y \subseteq \text{ran} \tau x \) is similar.

Thus \( \text{ran} \tau y = \text{ran} \tau x \) for all \( x, y, \eta \) such that \( \xi E_3 \eta \). Here we need another block of definitions.

Let \( H \) be the set of all \( m \in \mathbb{N} \times \mathbb{N} \) such that \( \delta = (m, j, m) \) is finite for all \( j \in \mathbb{N} \). For instance if \( \xi, \eta \in \mathbb{R}^\mathbb{N} \) satisfy \( \xi E_3 \eta \) then the set

\[
\delta_{xy} = \{ (j, m) : \xi(j)(m) \neq \eta(j)(m) \}
\]

belongs to \( H \). The operation of symmetric difference \( \Delta \) converts \( H \) into a Polish group equal to the product group \( \langle \mathbb{Z}^\mathbb{N} \rangle : \mathbb{Z}^N \Delta \mathbb{Z}^N \rangle \).

If \( n \in \mathbb{N} \), \( (a, \psi) \in D_n \), and \( \delta \in \mathbb{Z}^\mathbb{N} \) then we define a sequence \( \psi' = H_0^\delta(\psi) \in \mathcal{D}^n \) so that \( \psi' = (\delta \mid \xi(a)(m)) \Delta \psi(m) \) for every \( m < n \). Then the pair \( (a, H_0^\delta(\psi)) \) obviously still belongs to \( D_n \) and \( H_0^\delta(H_0^\delta(\psi)) = \psi \).

Coming back to a triple of \( y, \xi, \eta \in \mathbb{R}^\mathbb{N} \) such that \( \xi E_3 \eta \), let \( \delta = \delta_{xy} \). A routine verification shows that \( \tau y(a, \psi) = \tau y(a, H_0^\delta(\psi)) \) for all \( (a, \psi) \in D_n \). It follows that \( \tau y \subseteq \tau x \), as required. \( \square \) (Case 1 of Theorem 2)
It satisfies \( P_0^\prime \subseteq \Pi \) by Claim 17. Suppose that pairs \( \langle a, \varphi \rangle, \langle b, \psi \rangle \) belong to \( D_n \) for the same \( n \), and \( \langle x, \xi \rangle \in \mathbb{R}^N \times \mathbb{R}^N \). Put \( G_{\varphi \psi}^y (x, \xi) = (y, \xi) \in \mathbb{R}^N \times \mathbb{R}^N \), where

\[
\begin{align*}
y &\begin{cases}
\tau_y^x (b, \psi) & \text{whenever } x = \tau_x^y (a, \varphi), \\
\tau_y^x (a, \varphi) & \text{whenever } x = \tau_x^y (b, \psi), \\
x & \text{whenever } \tau_y^x (a, \varphi) \neq x \neq \tau_y^x (b, \psi).
\end{cases}
\end{align*}
\]

In our assumptions, \( y_{|} \geq n = x_{|} \geq n \) and \( G_{\varphi \psi}^y \) is a homeomorphism of \( \mathbb{R}^N \times \mathbb{R}^N \) onto itself and of \( \Pi \) onto itself, and \( G_{\varphi \psi}^y = G_{\psi \varphi}^x \). In addition we have \( \tau \circ \tau = \tau \circ \tau \) whenever \( (y, \xi) = a_{\varphi \psi} (x, \eta) \).

The group \( G \) of all superpositions of maps of the form \( G_{\varphi \psi}^y \), where \( \langle a, \varphi \rangle, \langle b, \psi \rangle \) belong to one and the same set \( D_n \), is a countable group of homeomorphisms of \( \mathbb{R}^N \times \mathbb{R}^N \). Consider the equivalence relation \( \Gamma \) induced by \( G \) on \( \Pi \). Thus \( \langle x, \xi \rangle \in \mathbb{R}^N \times \mathbb{R}^N \) iff there exists a homeomorphism \( g \in G \) such that \( g (x, \xi) = (y, \eta) \) (and then by definition \( \eta = \xi \)).

Now let us study relations between \( G \) and \( H \), the group introduced in the proof of Corollary 18. For any \( \delta \in \mathbb{H} \) define a homeomorphism \( H_\delta \) of \( \mathbb{R}^N \times \mathbb{R}^N \) so that \( H_\delta (x, \xi) = (x, \xi) \), where simply \( \eta = \delta \Delta \xi \) in the sense that

\[
\eta (m, j) = \begin{cases}
\xi (m, j) & \text{whenever } (m, j) \notin \delta, \\
1 - \xi (m, j) & \text{whenever } (m, j) \in \delta.
\end{cases}
\]

(Then obviously \( \delta = \delta \chi \)). If \( \gamma, \delta \in \mathbb{H} \) then the superposition \( H_\gamma \circ H_\delta = H_{\gamma \Delta \delta} \), where \( \Delta \) is the symmetric difference, as usual. Transformations of the form \( G_{\varphi \psi}^y \) do not commute with those of the form \( H_\delta \), yet there exists a convenient and easy to verify law of commutation:

**Lemma 20.** Suppose that \( n \in \mathbb{N} \) and pairs \( \langle a, \varphi \rangle, \langle b, \psi \rangle \) belong to \( D_n \), and \( \delta \in \mathbb{H} \). Then the superposition \( G_{\varphi \psi}^y \circ H_\delta \) coincides with \( H_\delta \circ G_{\psi \varphi}^y \), where \( \varphi' = H_\delta^a (\varphi) \) and \( \psi' = H_\delta^b (\psi) \).

It follows that the set \( S \) of all homeomorphisms \( s : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{R}^N \), of the form \( s = H_\delta \circ g_{\ell - 1} \circ g_{\ell - 2} \cdots \circ g_1 \circ g_0 \), where \( \ell \in \mathbb{N} \), \( \delta \in \mathbb{H} \), and each \( g_1 \) is a homeomorphism of \( \mathbb{R}^N \times \mathbb{R}^N \) of the form \( G_{\varphi \psi}^y \), and the pairs \( \langle a, \varphi \rangle, \langle b, \psi \rangle \) belong to one and the same set \( D_n \), \( n = n_1 \) (then \( g_{\ell - 1} \circ g_{\ell - 2} \cdots \circ g_1 \circ g_0 \in G \)), is a group under the superposition. For instance if \( g = G_{\varphi \psi}^y \) and \( g_1 \) belong to \( G \) and \( (a, \varphi) \), \( (b, \psi) \) belong to one and the same \( D_n \) then the superposition \( H_\delta \circ g \circ H_\beta \circ g_1 \) coincides with \( H_\delta \circ H_\beta \circ g' \circ g_1 = H_{\delta \Delta \beta} \circ (g' \circ g_1) \), where \( g' = G_{\psi \varphi}^y \) and \( \varphi' = H_\delta^b (\varphi) \), \( \psi' = H_\delta^b (\psi) \) as in Lemma 20.

Thus \( S \) is a more complicated product of \( G \) and \( H \), but on the other hand more elementary than the free product (of all formal superpositions of elements of both groups). The action of \( S \) on \( \mathbb{R}^N \times \mathbb{R}^N \) is defined as follows: if \( s \) is as above then \( s \cdot (x, \xi) = H_\delta (g_{\ell - 1} (g_{\ell - 2} (\cdots (g_1 (g_0 (x, \xi)) \cdots))) \). One can easily check that both the group \( S \) and the action are Polish. On the other hand, the induced orbit equivalence relation \( S \) is equal to the conjunction \( F \) of \( G \) and the equivalence relation \( E_3 \) acting on the 2nd factor of \( \mathbb{R}^N \times \mathbb{R}^N \), in the sense of Theorem 3 in the Introduction.

Moreover, we have \( \langle x, \xi \rangle \in E_3 (y, \eta) \) if and only if \( \langle x, \xi \rangle \in S \) for any \( \langle x, \xi \rangle, \langle y, \eta \rangle \in P_0^\prime \).

The final step is the next lemma. Its proof, not really obvious, see in [6].

**Lemma 21.** \( G \) is the union of an increasing sequence of finite subgroups, therefore the induced equivalence relation \( G \) is hyperfinite.

\[ \square \text{ (Theorem 3)} \]

The arguments above reduce further study of Case 1 of Theorem 2 to properties of the group \( S \) and its Polish actions. This is an open topic, and maybe the local finiteness of \( G \) (by Lemma 21) can lead to more comprehensive results.

7. Case 2

Then the \( \Sigma_1^1 \) set \( R = P_0 \cap H \), where \( H = 2^N \times S \) is the chaotic domain, is non-empty. Our goal will be to prove that \( E_1 \subseteq E_3 \mid R \) in this case. The embedding \( \vartheta : \mathbb{R}^N \to \mathbb{R} \) will have the property that any two elements \( \langle x, \xi \rangle \) and \( \langle x', \xi' \rangle \) in the range \( \vartheta \subseteq \mathbb{R}^N \) satisfy \( x \subseteq \xi ' \neq \xi' \), so that the \( \xi' \)-component in the range of \( \vartheta \) is trivial. And as far as the \( x \)-component is concerned, the embedding will resemble the embedding defined in Case 1 of the proof of the 1st dichotomy theorem in [10] (see also [8, Ch. 8]).

Recall that sets \( S_0^m \) were defined in Corollary 10, and by definition

\[
\begin{align*}
\langle x, \xi \rangle \in H \quad \iff \quad & \forall m \exists n \geq m \forall k \left( \langle x, \xi \rangle \notin S_0^m \right) \\
\iff \forall m \exists n \geq m \forall k \varphi \in S_0^m \left( \langle x, \xi \rangle \notin S_0^m \right)
\end{align*}
\]

in Case 2. Prove a couple of related technical lemmas.
Lemma 22. Each set $S^k_n$ is invariant in the following sense: if $(x, \xi) \in S^k_n$, $(y, \eta) \in \mathbb{R}^N \times \mathbb{R}^N$, $x|_\geq n = y|_\geq n$, and $\xi \vdash \eta$ then $(y, \eta) \in S^k_n$.

Proof. Otherwise there is a $\Delta^1_1$ function $\varphi \in \mathcal{P}_n^k$ such that $y(n) = \varphi(y|_\geq n, \eta)$. Then $x(n) = \varphi(x|_\geq n, \eta)$ as well because $x|_\geq n = y|_\geq n$. We put

$$u_j = \langle \xi(j), \Delta \eta(j) = \{m: \xi(j)(m) \neq \eta(j)(m)\}\rangle$$

for every $j < k$, these are finite subsets of $\mathbb{N}$. If $a \subseteq 2^N$ and $u \subseteq \mathbb{N}$ then define $u \cdot a \in 2^N$ so that $(u \cdot a)(m) = a(m)$ for $m \notin u$, and $(u \cdot a)(m) = a(m)$ for $m \in u$. If $\xi \in \mathbb{R}^N$ then define $f(\xi) \in \mathbb{R}^N$ so that $f(\xi)(j) = u_j \cdot \xi(j)$ for $j < k$, and $f(\xi)(j) = \xi(j)$ for $j \geq k$.

Finally, put $\psi(z, \xi) = \varphi(z, f(\xi))$ for every $(z, \xi) \in \mathbb{R}^N \times \mathbb{R}^N$. The map $\psi$ obviously belongs to $\mathcal{P}_n^k$ together with $\varphi$. Moreover

$$\psi(\xi(n)) = \varphi(x|_\geq n, f(\eta)) = \psi(x|_\geq n, \xi)$$

because $f(\eta)|_\leq k = \xi|_\leq k$, and this contradicts to the choice of $(x, \xi)$. □

The next simple lemma will allow us to split $\Sigma^1_1$ sets in $\mathbb{R}^N \times \mathbb{R}^N$.

Lemma 23. If $P \subseteq \mathbb{R}^N \times \mathbb{R}^N$ is a $\Sigma^1_1$ set and $P \notin S^k_n$ then there exist points $(x, \xi)$ and $(y, \eta)$ in $P$ with

$$y|_\geq n = x|_\geq n, \quad \eta \vdash \xi, \quad \eta|_\leq k = \xi|_\leq k, \quad \text{but} \quad y(n) \neq x(n).$$

Proof. Otherwise $\psi = \{((y|_\geq n, \eta) \in y(n)): (y, \eta) \in P\}$ is a map in $\mathcal{P}_n^k$, and hence $P \subseteq S^k_n$, contradiction. □

8. Case 2: splitting system

We apply a splitting construction, developed in [5] for the study of “ill”founded Sacks iterations. Below, $2^N$ will typically denote the set of all dyadic sequences of length $n$, and $2^{<\omega} = \bigcup_n 2^n$ all finite dyadic sequences.

The construction involves a map $\varphi : \mathbb{N} \to \mathbb{N}$ assuming infinitely many values and each its value infinitely many times (but $\forall n \varphi$ may be a proper subset of $\mathbb{N}$), another map $\pi : \mathbb{N} \to \mathbb{N}$, and, for each $u \in 2^{<\omega}$, a non-empty $\Sigma^1_1$ subset $P_u \subseteq R \cap H$ which satisfies a quite long list of properties.

First of all, if $\varphi$ is already defined at least on $0, n$ and $u \neq v \in 2^n$ then let $\nu_u[v, u] = \max \{\varphi(\ell): \ell < n \land u(\ell) \neq v(\ell)\}$. And put $\nu_u[u, u] = -1$ for any $u$.

Now we present the list of requirements $1^\circ-8^\circ$.

$1^\circ$: if $\varphi(n) \notin \{\varphi(\ell): \ell < n\}$ then $\varphi(n) > \varphi(\ell)$ for each $\ell < n$;

$2^\circ$: if $u \in 2^n$ then $P_u \cap (\bigcup_{n} S^k_u) = \emptyset$ for each $\ell < n$;

$3^\circ$: every $P_u$ is a non-empty $\Sigma^1_1$ subset of $R \cap H$;

$4^\circ$: $P_{u|\ell} \subseteq P_u$ for all $u \in 2^{<\omega}$ and $i = 0, 1$.

Two further conditions are related rather to the sets $X_u = \text{dom} P_u$.

$5^\circ$: if $u, v \in 2^n$ then $X_u|_{\nu_u[u, v]} = X_v|_{\nu_v[u, v]}$;

$6^\circ$: if $u, v \in 2^n$ then $X_u|_{\nu_u[u, v]} \cap X_v|_{\nu_v[u, v]} = \emptyset$.

The content of the next condition is some sort of genericity in the sense of the Gandy–Harrington forcing in the space $\mathbb{R}^N \times \mathbb{R}^N$, that is, the forcing notion

$$\mathbb{P} = \{\text{all non-empty } \Sigma^1_1 \text{ subsets of } \mathbb{R}^N \times \mathbb{R}^N\}.$$ 

Let us fix a countable transitive model $M$ of a sufficiently large fragment of ZFC.\footnote{For instance remove the Power Set axiom but add the axiom saying that for any set $X$ there exists the set of all countable subsets of $X$.} For technical reasons, we assume that $M$ is an elementary submodel of the universe w.r.t. all analytic formulas. Then simple relations between sets in $\mathbb{P}$ in the universe, like $P = Q$ or $P \subseteq Q$, are adequately reflected as the same relations between their intersections $P \cap M, Q \cap M$ with the model $M$. In this sense $\mathbb{P}$ is a forcing notion in $M$.

A set $D \subseteq \mathbb{P}$ is open dense iff, first, for any $P \in \mathbb{P}$ there is $Q \subseteq D, Q \subseteq P$, and given sets $S \subseteq Q \in \mathbb{R}$, if $Q$ belongs to $D$ then so does $P$. A set $D \subseteq \mathbb{P}$ is coded in $M$, iff the set $\{P \cap M: P \in D\}$ belongs to $M$. There exists at most countably many such sets because $M$ is countable. Let us fix an enumeration (not in $M$) $\{D_n: n \in \mathbb{N}\}$ of all open dense sets $D \subseteq \mathbb{P}$ coded in $M$.
The next condition essentially asserts the \( P \)-genericity of each branch in the splitting construction over \( M \).

\[ 7^\circ: \text{for every } n, \text{ if } u \in 2^{n+1} \text{ then } P_u \in D_n. \]

**Remark 24.** It follows from \( 7^\circ \) that for any \( a \in 2^N \) the sequence \( \{P_a\}_{a \in N} \) is generic enough for the intersection \( \bigcap_n P_a \neq \emptyset \) to consist of a single point, say \( (g(a), \gamma(a)) \), and for the maps \( g, \gamma : 2^N \to \mathbb{R}^N \times \mathbb{R}^N \) to be continuous.

Note that \( g = 1 - 1\). Indeed if \( a \neq b \) belong to \( 2^N \) then \( a(n) \neq b(n) \) for some \( n \), and hence \( \nu \psi(a \upharpoonright m, b \upharpoonright m) \geq \psi(n) \) for all \( m \geq n \). It follows by \( 6^\circ \) that \( X_a \cap X_b = \emptyset \) for \( m > n \), therefore \( g(a) \neq g(b) \).

Our final requirement involves the \( \xi \)-parts of sets \( P_u \). We’ll need the following definition. Suppose that \( (x, \xi) \) and \( (y, \eta) \) belong to \( \mathbb{R}^N \times \mathbb{R}^N \), \( p \in \mathbb{N} \), and \( s \in \mathbb{N}^{<\omega} \), \( 1hs = m \) (the length of \( s \)). Define \( (x, \xi) \equiv_p (y, \eta) \) iff

\[ \xi \in E_3 \eta, \quad x|_p = y|_p, \quad \text{and} \quad \xi(k) \Delta \eta(k) \subseteq s(k) \quad \text{for all } k < m = 1hs, \]

where \( \alpha \Delta \beta = \{ j : \alpha(j) \neq \beta(j) \} \) for \( \alpha, \beta \in 2^N \). If \( P, Q \subseteq \mathbb{R}^N \times \mathbb{R}^N \) are arbitrary sets then under the same circumstances \( P \equiv_p Q \) will mean that

\[ \forall (x, \xi) \in P \exists (y, \eta) \in Q \ ( (x, \xi) \equiv_p (y, \eta) ) \quad \text{and vice versa}. \]

Obviously \( \equiv_1^p \) is an equivalence relation.

The following is the last condition:

\[ 8^\circ: \text{there exists a map } \pi : N \to N, \text{ such that } P_u \equiv_{\pi|_n}^\nu \nu \psi[u, v] \forall v \forall v \in 2^N \quad \text{and then} \quad X_u \cap \nu \psi[u, v] = X_v \cap \nu \psi[u, v] \quad \text{as in } 5^\circ. \]

**9. Case 2: splitting system implies the reducibility**

Here we prove that any system of sets \( P_u \) and \( X_u = \text{dom} P_u \) and maps \( \psi, \pi \) satisfying \( 1^\circ - 8^\circ \) implies Borel reducibility of \( E_1 \) to \( E_{13} \mid R \). This completes Case 2. The construction of such a splitting system will follow in the remainder.

Let the maps \( g \) and \( \gamma \) be defined as in Remark 24. Put

\[ W = \{ (g(a), \gamma(a)) : a \in 2^N \}. \]

**Lemma 25.** \( W \) is a closed set in \( \mathbb{R}^N \times \mathbb{R}^N \) and a function. Moreover if \( (x, \xi) \) and \( (y, \eta) \) belong to \( W \) then \( \xi \in E_3 \eta \).

**Proof.** \( W \) is closed as a continuous image of \( 2^N \). That \( W \) is a function follows from the bijectivity of \( g \), see Remark 24. Finally any two \( \xi, \eta \) as indicated satisfy \( \xi(k) \Delta \eta(k) \subseteq \pi(k) \) for all \( k \) by \( 8^\circ \). \( \square \)

Put \( X = \text{dom} W \). Thus \( W \) is a continuous map \( X \to \mathbb{R}^N \) by the lemma.

**Corollary 26.** There exists a Borel reduction of \( E_{1} \mid X \) to \( E_{13} \mid W \).

**Proof.** As \( W \) is a function, we can use the notation \( W(x) \) for \( x \in X = \text{dom} W \). Put \( f(x) = (x, W(x)) \). This is a Borel, even a continuous map \( X \to W \). It remains to establish the equivalence

\[ x E_1 y \iff f(x) E_{13} f(y) \quad \text{for all } x, y \in X. \tag{6} \]

If \( x E_1 y \) then \( W(x) E_3 W(y) \) by Lemma 25, and hence easily \( f(x) E_{13} f(y) \). If \( x E_1 y \) fails then obviously \( f(x) E_{13} f(y) \) fails, too. \( \square \)

Thus to complete Case 2 it now suffices to define a Borel reduction of \( E_{1} \) to \( E_{1} \mid X \). To get such a reduction consider the set \( \Phi = r \times \text{dom} \psi \), and let \( \Phi = \{ p_m : m \in \mathbb{N} \} \) in the increasing order; that the set \( \Phi \subseteq \mathbb{N} \) is infinite follows from \( 1^\circ \).

Suppose that \( n \in \mathbb{N} \). Then \( \psi(n) = p_m \) for some (unique) \( m \); we put \( \psi(n) = m \). Thus \( \psi : \mathbb{N}^{<\omega} \to \mathbb{N} \) and the preimage \( \psi^{-1}(m) = \psi^{-1}(p_m) \) is an infinite subset of \( N \) for any \( m \). Define a parallel system of sets \( Y_u \subseteq \mathbb{R}^N \), \( u \in 2^{<\omega} \), as follows. Put \( Y_0 = \mathbb{R}^N \). Suppose that \( Y_u \) has been defined, \( u \in 2^{<\omega} \). Put \( p = \psi(n) = p_{\psi(n)} \). Let \( K \) be the number of all indices \( \ell < n \) still satisfying \( \psi(\ell) = p \), perhaps \( K = 0 \). Put \( Y_u + 1 = \{ x \in Y_u : \psi(p(K - 1)) = \ell \} \) for \( i = 0, 1 \).

Each of \( Y_u \) is clearly a basic clopen set in \( \mathbb{R}^N \), and one easily verifies that conditions \( 4^\circ - 6^\circ \) are satisfied for the sets \( Y_u \) and the map \( \psi \) (instead of \( \psi \) in \( 5^\circ, 6^\circ \)), in particular

\[ 6^\circ: \text{if } u, v \in 2^N \text{ then } Y_u \upharpoonright \psi[u, v] = Y_v \upharpoonright \psi[u, v]; \]

\[ 7^\circ: \text{if } u, v \in 2^N \text{ then } Y_u \upharpoonright \psi[u, v] \cap Y_v \upharpoonright \psi[u, v] = \emptyset; \]

where \( \nu \psi[u, v] = \max \{ \psi(\ell) : \ell < u \land u(\ell) \neq v(\ell) \} \) (compare with \( \psi \) above).
It is clear that for any \( a \in 2^\mathbb{N} \) the intersection \( \bigcap u Y_\mathfrak{a}[n] = \{ f(a) \}\) is a singleton, and the map \( f \) is continuous and 1–1. (We can, of course, define \( f \) explicitly: \( f(a)(p)(K) = a(n) \), where \( n \in \mathbb{N} \) is chosen so that \( \psi(n) = p \) and there is exactly \( K \) numbers \( \ell < n \) with \( \psi(\ell) = p \).) Note finally that \( \{ f(a) : a \in 2^\mathbb{N} \} = \mathbb{R}^\mathbb{N} \) since by definition \( Y_{\mathfrak{a}+1} \cup Y_{\mathfrak{a}+0} = Y_{\mathfrak{a}} \) for all \( \mathfrak{a} \).

We conclude that the map \( \vartheta(x) = g(f^{-1}(x)) \) is a continuous map (in fact a homeomorphism in this case by compactness) \( \mathbb{R}^\mathbb{N} \xrightarrow{\vartheta} X = \text{dom} W \).

**Lemma 27.** The map \( \vartheta \) is a reduction of \( E_1 \) to \( E_1 \mid X \), and hence \( \vartheta \) witnesses \( E_1 \leq^B E_1 \mid X \) and \( E_1 \leq^B E_{13} \mid W \) by Corollary 26.

**Proof.** It suffices to check that the map \( \vartheta \) satisfies the following requirement: for each \( y, y' \in \mathbb{R}^\mathbb{N} \) and \( m \),

\[
y \mid \geq m = y' \mid \geq m \iff \vartheta(y) \mid \geq pm = \vartheta(y') \mid \geq pm. \tag{7}
\]

To prove (7) suppose that \( y = f(a) \) and \( x = g(a) = \vartheta(y) \), and similarly \( y' = f(a') \) and \( x' = g(a') = \vartheta(y') \), where \( a, a' \in 2^\mathbb{N} \). Suppose that \( y \mid \geq m = y' \mid \geq m \). We then have \( m > \nu(y)[n, a' \mid n] \) for any \( n \) by \( 7^\circ \). It follows, by the definition of \( \psi \), that \( \nu \mid \geq m \) is continuous and 1–1. Hence, \( X_{a \mid n} \mid \geq pm = X_{a' \mid n} \mid \geq pm \) for any \( n \) by \( 5^\circ \). Therefore \( x \mid \geq pm = x' \mid \geq pm \) by \( 7^\circ \), that is, the right-hand side of (7). The inverse implication in (7) is proved similarly.

It follows that we can now focus on the construction of a system satisfying \( 1^\circ – 8^\circ \). The construction follows in Section 12, after several preliminary lemmas in Sections 10 and 11.

10. Case 2: how to shrink a splitting system

Let us prove some results related to preservation of condition \( 8^\circ \) under certain transformations of shrinking type. They will be applied in the construction of a splitting system satisfying conditions \( 1^\circ – 8^\circ \) of Section 8.

**Lemma 28.** Suppose that \( n \in \mathbb{N} \), \( s \in \mathbb{N} \), and a system of \( \Sigma^1_1 \) sets \( \emptyset \neq P_u \subseteq \mathbb{R}^\mathbb{N} \times \mathbb{R}^\mathbb{N} \), \( u \in 2^n \), satisfies \( P_u \cong^s v_{w, u} P_\nu \) for all \( u, v \in 2^n \). Assume also that \( w_0 \in 2^n \), and \( \emptyset \neq Q \subseteq P_{w_0} \) is a \( \Sigma^1_1 \) set. Then the system of \( \Sigma^1_1 \) sets \( P'_u = \{ (x, \xi) \in P_u : \exists (z, \zeta) \in Q \langle (x, \xi) \cong^s v_{w, w_0} (z, \zeta) \rangle, \ u \in 2^n \}, \) still satisfies \( P'_u \cong^s v_{w, u} P'_\nu \) for all \( u, v \in 2^n \), and \( P'_{w_0} = Q \).

**Proof.** \( P'_{w_0} = Q \) holds because \( v_{w_0}[w_0, w_0] = -1 \). Let us verify \( 8^\circ \). Suppose that \( u, v \in 2^n \). Each one of the three numbers \( v_{w_0}[u, v], v_{w_0}[v, w], v_{w_0}[u, v] \) is obviously not bigger than the largest of the two other numbers. This observation leads us to the following three cases.

Case a: \( v_{w_0}[u, w] = v_{w_0}[v, v] \geq v_{w_0}[u, v] \). Consider any \( (x, \xi) \in P'_u \). Then by definition there exists \( (z, \zeta) \in Q \) with \( (x, \xi) \cong^s v_{w, w_0} (z, \zeta) \). Then, as \( P'_u \cong^s v_{w, w_0} P_\nu \) is assumed by the lemma, there is \( (y, \eta) \in P_\nu \) such that \( (y, \eta) \cong^s v_{w, w_0} (z, \zeta) \). Note that \( (z, \zeta) \) witnesses \( (y, \eta) \in P'_\nu \). On the other hand, \( (x, \xi) \cong^s v_{w, w_0} (y, \eta) \) because \( v_{w_0}[u, w_0] = v_{w_0}[v, v] \geq v_{w_0}[w, w_0] \). Conversely, suppose that \( (y, \eta) \in P'_\nu \). Then there is \( (z, \zeta) \in Q \) such that \( (y, \eta) \cong^s v_{w_0} (z, \zeta) \). Yet \( P_{w_0} \cong^s v_{w_0} P_u \), and hence there exists \( (x, \xi) \in P'_u \) with \( (x, \xi) \cong^s v_{w_0} (z, \zeta) \). Once again we conclude that \( (x, \xi) \cong^s v_{w_0} (y, \eta) \).

Case b: \( v_{w_0}[v, w] = v_{w_0}[u, v] \geq v_{w_0}[u, w] \). Absolutely similar to Case a.

Case c: \( v_{w_0}[w, w_0] = v_{w_0}[v, v] \geq v_{w_0}[u, v] \). This is a symmetric case, thus it is enough to carry out only the direction \( P'_u \rightarrow P'_\nu \). Consider any \( (x, \xi) \in P'_u \). As above there is \( (z, \zeta) \in Q \) such that \( (x, \xi) \cong^s v_{w_0} (z, \zeta) \). On the other hand, as \( P'_u \cong^s v_{w, w_0} P_\nu \), there exists a point \( (y, \eta) \in P_\nu \) such that \( (y, \eta) \cong^s v_{w, w_0} (x, \xi) \). Note that \( (x, \xi) \) witnesses \( (y, \eta) \in P'_\nu \); indeed by definition we have \( (y, \eta) \cong^s v_{w_0} (x, \xi) \).

**Corollary 29.** Assume that \( n \in \mathbb{N} \), \( s \in \mathbb{N} \), and a system of \( \Sigma^1_1 \) sets \( \emptyset \neq P_u \subseteq \mathbb{R}^\mathbb{N} \times \mathbb{R}^\mathbb{N} \), \( u \in 2^n \), satisfies \( P_u \cong^s v_{w, u} P_\nu \) for all \( u, v \in 2^n \). Assume also that \( \emptyset \neq Q \subseteq P_{w_0} \) is defined for every \( w \in W \) so that still \( Q_w \cong^s v_{w, w_0} Q_\nu \) for all \( w, w' \in W \). Then the system of \( \Sigma^1_1 \) sets \( P'_u = \{ (x, \xi) \in P_u : \exists w \in W \exists (y, \eta) \in Q_w \langle (x, \xi) \cong^s v_{w} (y, \eta) \rangle \} \) still satisfies \( P'_u \cong^s v_{w, u} P'_\nu \) for all \( u, v \in 2^n \), and \( P'_{w_0} = Q_w \) for all \( w \in W \).

**Proof.** Apply the transformation of Lemma 28 consecutively for all \( w_0 \in W \) and the corresponding sets \( Q_{w_0} \). Note that these transformations do not change the sets \( Q_w \) with \( w \in W \) because \( Q_w \cong^s v_{w, w_0} Q_{w'} \) for all \( w, w' \in W \).
Remark 30. The sets $P'_u$ in Corollary 29 can as well be defined by
\[ P'_u = \{ (x, \xi) \in P_u : \exists \eta \in Q_w \ (x, \xi) \equiv^s_{v_p[u,w]} (y, \eta) \} \]
where, for each $u \in 2^n$, $w_u$ is an element of $W$ such that the number $v_p[u, w_u]$ is the least of all numbers of the form $v_p[u, w]$, $w \in W$. (If there exist several $w \in W$ with the minimal $v_p[u, w]$ then take the least of them.)

11. Case 2: how to split a splitting system

Here we consider a different question related to the construction of systems satisfying conditions 1°–8° of Section 8. Given a system of $\Sigma^1_1$ sets satisfying a 8°-like condition, how to shrink the sets so that 8° is preserved and in addition 6° holds. Let us begin with a basic technical question: given a pair of $\Sigma^1_1$ sets $P, Q$ satisfying $P \equiv^s_p Q$, for some $p, s$, how to define a pair of smaller $\Sigma^1_1$ sets $P' \subseteq P$, $Q' \subseteq Q$, still satisfying the same condition, but as disjoint as it is compatible with this condition.

Recall that $\text{dom} P = \{ x : \exists \xi \ (x, \xi) \in P \}$ for $P \subseteq \mathbb{R}^N \times \mathbb{R}^N$.

Lemma 31. If $P, Q \subseteq \mathbb{R}^N \times \mathbb{R}^N$ are non-empty $\Sigma^1_1$ sets, $p, n, s \in \mathbb{N}^\omega$, $P \equiv^p_n Q$, and $(P \cup Q) \cap s_k = \emptyset$, where $k = 1, 2, \ldots, n$, then there exist non-empty $\Sigma^1_1$ sets $P' \subseteq P$, $Q' \subseteq Q$ such that $P' \equiv^p_n Q'$, but in addition $(\text{dom} P') \supseteq_p (\text{dom} Q')$.

Note that $P \equiv^p_n Q$ implies $(\text{dom} P)|_{>p} = (\text{dom} Q)|_{>p}$.

Proof. It follows from Lemma 23 that there exist points $(x_0, \xi_0)$ and $(x_1, \xi_1)$ in $P$ such that $(x_0, \xi_0) \equiv^s_p (x_1, \xi_1)$ but $x_1(p) \neq x_0(p)$). Then there exists a number $j$ such that, say, $x_1(p)(j) = 1 \neq x_0(p)(j)$. On the other hand, there exists $(y_0, \eta_0) \in Q$ such that $(x_i, \xi_i) \equiv^p_j (y_0, \eta_0)$ for $i = 0, 1$. Then $y_0(p)(j) \neq x_0(p)(j)$ for some $i = 0, 1$. Let say $y_0(p)(j) = 0 \neq x_0(p)(j)$ whenever $(x, \xi) \in P'$ and $(y, \eta) \in Q'$.

Corollary 32. Assume that $n \in \mathbb{N}$, $s \in \mathbb{N}^\omega$, and a system of $\Sigma^1_1$ sets $\emptyset \neq P_u \subseteq \mathbb{R}^N \times \mathbb{R}^N$, $u \in 2^n$, satisfies $P_u \equiv^s_{v_p[u,v]} P_v$ for all $u, v \in 2^n$. Then there exists a system of $\Sigma^1_1$ sets $\emptyset \neq P'_u \subseteq P_u$, $u \in 2^n$, such that $P'_u \equiv^s_{v_p[u,v]} P_v$, and in addition $(\text{dom} P'_u)|_{>v_p[u,v]} \cap (\text{dom} P_v)|_{>v_p[u,v]} = \emptyset$, for all $u \neq v \in 2^n$.

Proof. Consider any pair of $u_0 \neq v_0$ in $2^n$. Apply Lemma 31 for the sets $P = P_{u_0}$ and $Q = P_{v_0}$ and $p = v_p[u_0, v_0]$. Let $P'$ and $Q'$ be the $\Sigma^1_1$ sets obtained, in particular $P' \equiv^s_{v_p[u_0, v_0]} Q'$ and $(\text{dom} P')|_{>v_p[u_0, v_0]} \cap (\text{dom} Q')|_{>v_p[u_0, v_0]} = \emptyset$. Then by Corollary 29 there is a system of $\Sigma^1_1$ sets $\emptyset \neq P'_u \subseteq P_u$ such that $P'_u \equiv^s_{v_p[u,v]} P_v'$ for all $u, v \in 2^n$, and $P_{u_0} = P'$, $P_{v_0} = Q'$, and hence $(\text{dom} P'_u)|_{>v_p[u,v_0]} \cap (\text{dom} P_{v_0}'|_{>v_p[u,v_0]} = \emptyset$.

Take any other pair of $u_1 \neq v_1$ in $2^n$ and transform the system of sets $P'_u$ the same way. Iterate this construction sufficient (finite) number of steps.

12. Case 2: the construction of a splitting system

We continue the proof of Theorem 2 — Case 2. Recall that $R = P_0 \cap H$ is a $\Sigma^1_1$ set. By Lemma 27, it suffices to define functions $\varphi$ and $\pi$ and a system of $\Sigma^1_1$ sets $P_u \subseteq R$ together satisfying conditions 1°–8°. The construction of such a system will go on by induction on $n$. That is, at any step $n$ the sets $P_u$ with $u \in 2^n$, as well as the values of $\varphi(k)$ and $\pi(k)$ with $k < n$, will be defined.

For $n = 0$, we put $P_A = R$. ($A \in 2^0$ is the only sequence of length 0.)

Suppose that sets $P_u \subseteq R$ with $u \in 2^n$, and also all values $\varphi(\ell)$, $\ell < n$, and $\pi(k)$, $k < n$, have been defined and satisfy the applicable part of 1°–8°. The content of the inductive step $n \mapsto n + 1$ will consist in definition of $\varphi(n)$, $\pi(n)$, and sets $P_u^{n+1}$ with $u^{n+1}$ in $2^{n+1}$, that is, $u \in 2^n$ (a dyadic sequence of length $n$) and $i = 0, 1$. This goes on in four steps A, B, C, D.
12.1. Step A: definition of $\varphi(n)$

Suppose that, in the order of increase,
\[
\{\varphi(\ell) : \ell < n\} = \{p_0 < \cdots < p_m\}.
\]
For $j \leq m$, let $K_j$ be the number of all $\ell < n$ with $\varphi(\ell) = p_j$.

Case A: $K_j \geq m$ for all $j \leq m$. Then consider any $u_0 \in 2^n$ and an arbitrary point $\langle x_0, \xi_0 \rangle \in P_{u_0}$. Note that by (5) of Section 7 there is a number $p > \max_{\ell < n} \varphi(\ell)$ such that $(x_0, \xi_0) \notin \bigcup_{k} S_{\varphi(\ell)}^k$. Put $\varphi(n) = p$.

We claim that the sets $P_u' = P_u \setminus \bigcup_{k} S_{\varphi(n)}^k$ still satisfy condition $8^\circ$ (and then $5^\circ$ for $X'_u = \compl P_u'$). Indeed suppose that $u, v \in 2^n$ and $\langle x, \xi \rangle \in P_u'$. Then $\langle x, \xi \rangle \in P_u$, and hence there is a point $\langle y, \eta \rangle \in P_v$ such that $\langle x, \xi \rangle \equiv^\varphi[n]_{v[u,v]} \langle y, \eta \rangle$. It remains to show that $\langle y, \eta \rangle \notin \bigcup_{k} S_{\varphi(n)}^k$. Suppose towards the contrary that $\langle y, \eta \rangle \in S_{\varphi(n)}^k$ for some $k$. By definition $\varphi(n) > v[y, u, v]$, therefore $\langle x, \xi \rangle \in S_{\varphi(n)}^k$ by Lemma 22, contradiction.

Case B: If some numbers $K_j$ are $m$ then choose $\varphi(n)$ among $p_j$ with the least $K_j$, and among them take the least one. Thus $\varphi(n) = \varphi(\ell)$ for some $\ell < n$. It follows that in this case $P_u \cap (\bigcup_{k} S_{\varphi(n)}^k) = \emptyset$ for all $u \in 2^n$ by the inductive assumption of $2^\circ$. Put $P_u' = P_u$.

Note that this manner of choice of $\varphi(n)$ implies $1^\circ, 2^\circ$ and also implies that $\varphi$ takes infinitely many values and takes each its value infinitely many times. In addition, the construction given above proves:

**Lemma 33.** There exists a system of $\Sigma^1_1$ sets $\emptyset \neq P_u' \subseteq P_u$ satisfying $8^\circ$ and $P_u' \cap (\bigcup_{k} S_{\varphi(n)}^k) = \emptyset$ for all $u \in 2^n$.

12.2. Step B: definition of $\pi(n)$

We work with the sets $P_u'$ such as in Lemma 33. The next goal is to prove the following result:

**Lemma 34.** There exist a number $r \in \mathbb{N}$ and a system of $\Sigma^1_1$ sets $\emptyset \neq P_u'' \subseteq P_u'$ satisfying $P_u'' \equiv^\varphi[n]_{v[u,v]} P_v''$ for all $u, v \in 2^n$.

**Proof.** Let $2^n = \{u_j : j < K\}$ be an arbitrary enumeration of all dyadic sequences of length $n$; $K = 2^n$, of course. The method of proof will be to define, for any $k \leq K$, a number $r_k \in \mathbb{N}$ and a system of $\Sigma^1_1$ sets $\emptyset \neq Q^k_{u_j} \subseteq P_{u_j}$, $j < k$, by induction on $k$ so that

\[
\langle x, \xi \rangle \equiv^\varphi[n]_{v[u_0,u_1]} Q^k_{u_j} \quad \text{for all } i < j < k.
\]

(Where $(\pi \upharpoonright n)^k$ is the extension of the finite sequence $\pi \upharpoonright n$ by $r$ as the new rightmost term.)

After this is done, $r = r_K$ and the sets $P_u'' = Q^K_{v[u_0,u_1]}$ prove the lemma.

We begin with $k = 2$. Then $P_{u_0} \equiv^\varphi[n]_{v[u_0,u_1]} P_{u_1}$ by $8^\circ$, and hence there exist points $\langle x_0, \xi_0 \rangle \in P_{u_0}$, $\langle x_1, \xi_1 \rangle \in P_{u_1}$ such that $\langle x_0, \xi_0 \rangle \equiv^\varphi[n]_{v[u_0,u_1]} \langle x_1, \xi_1 \rangle$. Then $\xi_0 \equiv_1 \xi_1$, and so there is a number $r \in \mathbb{N}$ with $\xi_0(n) \Delta \xi_1(n) \subseteq r_2$. Note that for any $p \in \mathbb{N}$ and any points $\langle x, \xi \rangle, \langle y, \eta \rangle \in \mathbb{R}^n \times \mathbb{R}^n$, $\langle x, \xi \rangle \equiv^\varphi[n]_{v[u_0,u_1]} \langle y, \eta \rangle$ is equivalent to the conjunction

\[
\langle x, \xi \rangle \equiv^\varphi[n]_{v[u_0,u_1]} \langle y, \eta \rangle \wedge \xi(n) \Delta \eta(n) \subseteq r.
\]

It follows that the sets

\[
S_0 = \left\{ \langle x, \xi \rangle \in P_{u_0}' : \exists \langle y, \eta \rangle \in P_{u_1}' \left( \langle x, \xi \rangle \equiv^\varphi[n]_{v[u_0,u_1]} \langle y, \eta \rangle \right) \right\}
\]

and

\[
S_1 = \left\{ \langle y, \eta \rangle \in P_{u_1}' : \exists \langle x, \xi \rangle \in P_{u_0}' \left( \langle x, \xi \rangle \equiv^\varphi[n]_{v[u_0,u_1]} \langle y, \eta \rangle \right) \right\}
\]

are $\Sigma^1_1$ and non-empty (contain resp. $(x_0, \xi_0)$ and $(x_1, \xi_1)$), and they obviously satisfy $S_0 \equiv^\varphi[n]_{v[u_0,u_1]} S_1$. Therefore by Corollary 29 there exists a system of $\Sigma^1_1$ sets $\emptyset \neq Q^2_{u_2} \subseteq P_{u_2}$, $u \in 2^n$, such that $Q^2_{u_0} = S_0$, $Q^2_{u_1} = S_1$, $8^\circ$ still holds, and in addition $Q^2_{u_0} \equiv^\varphi[n]_{v[u_0,u_1]} Q^2_{u_1}$. Put $r_2 = r$.

Now let us carry out the step $k \mapsto k + 1$. Suppose that $r_k$ and sets $Q^k_{u_j}$, $j < k$, satisfy $\ast$. Of all numbers $v[p_{u_j}, u_k]$, $j < k$, consider the least one. Let this be, say, $v[p_{u_j}, u_k]$, so that $j < k$ and $v[p_{u_j}, u_k] \leq v[p_{u_j}, u_j]$ for all $j < k$. As above there exists a number $r$ and a pair of non-empty $\Sigma^1_1$ sets $S_k \subseteq Q^k_{u_k}$ and $S_k \subseteq Q^k_{u_k}$ such that $S_e \equiv^\varphi[n]_{v[u_k,u_j]} S_k$. We can assume that $r \geq r_k$. Put

\[
Q'_{u_j} = \left\{ \langle y, \eta \rangle \in S_{u_j} : \exists \langle x, \xi \rangle \in S_k \left( \langle x, \xi \rangle \equiv^\varphi[n]_{v[u_k,u_j]} \langle y, \eta \rangle \right) \right\}
\]
for all \( j < k \). The proof of Lemma 28 shows that \( Q'_{uk} \) are non-empty \( \Sigma_1 \) sets still satisfying (\( *) \) in the form of
\[
Q'_{uj} \equiv (\pi[n])^r_{\nu\phi} Q'_{uk} \text{ for } i < j < k \quad \text{since } r \geq r_j \text{ and obviously } Q'_{uj} = S_r.
\]
In addition, put \( Q'_{uk} = S_k \). Then still \( Q'_{uk} \equiv (\pi[n])^r_{\nu\phi} Q'_{uk} \) by the choice of \( S_r \) and \( S_k \). We claim that also
\[
Q'_{uj} \equiv (\pi[n])^r_{\nu\phi} Q'_{uk} \quad \text{for all } j < k.
\]
Indeed we have \( Q'_{uj} \equiv (\pi[n])^r_{\nu\phi} Q'_{uk} \) and \( Q'_{uj} \equiv (\pi[n])^r_{\nu\phi} Q'_{uk} \) by the above. It follows that \( Q'_{uj} \equiv (\pi[n])^r_{\nu\phi} Q'_{uk} \), where \( p = \max \{v_p[u_j, u_k], v_p[u_j, u_k] \} \). Thus it remains to show that \( p \leq v_p[u_j, u_k] \). That \( v_p[u_j, u_k] \leq v_p[u_j, u_k] \) holds by the choice of \( \ell \). Prove that \( v_p[u_j, u_k] \leq v_p[u_j, u_k] \). Indeed in any case
\[
v_p[u_j, u_k] \leq \max \{v_p[u_j, u_k], v_p[u_j, u_k] \}.
\]
But once again \( v_p[u_j, u_k] \leq v_p[u_j, u_k] \), so \( v_p[u_j, u_k] \leq v_p[u_j, u_k] \) as required.

Thus (8) is established. It follows that \( Q'_{uj} \equiv (\pi[n])^r_{\nu\phi} Q'_{uj} \) for all \( i < j \leq k \). We end the inductive step of the lemma by putting \( r_{k+1} = r \).

12.3. Step C: splitting to the next level

We work with the number \( r \) and sets \( P''_u \) such as in Lemma 34. Put \( \pi(n) = r \). (Recall that \( \phi(n) \) was defined at Step A.)

The next step is to split each one of the sets \( P''_u \) in order to define sets \( P''_{u^n} \) in \( 2^{\aleph_0} \), of the next splitting level.

To begin with, put \( Q''_{u^n} = P''_u \) for all \( u \in 2^n \) and \( i = 0, 1 \). It is easy to verify that the system of sets \( Q''_{u^n} \), \( u^n \in 2^{\aleph_0} \), satisfies conditions 1° – 8° for the level \( n + 1 \), except for 7° and 6°. In particular, 2° was fixed at Step A, and 8° in the form that \( Q''_{u^n} \equiv (\pi[n])^r_{\nu\phi} Q''_{u^n} \) for all \( u^n \) and \( v^n \) in \( 2^{\aleph_0} \) (and then 5° as well) at Step B – because \( (\pi[n])^r_{\nu\phi} = \pi[n] \). Recall that by definition all sets involved have no common point with \( \bigcup_{n} S_{\phi(n)} \) by 2°. Therefore Corollary 32 is applicable. We conclude that there exists a system of non-empty \( \Sigma_1 \) sets \( W_{u^n} \subseteq Q''_{u^n} \), \( u^n \in 2^{\aleph_0} \), still satisfying 8°, and also satisfying 6°.

12.4. Step D: genericity

We have to further shrink the sets \( W_{u^n} \), \( u^n \in 2^{\aleph_0} \), obtained at Step C, in order to satisfy 7°, the last condition not yet fulfilled in the course of the construction. The goal is to define a new system of \( \Sigma_1 \) sets \( W_{u^n} \subseteq W_{u^n} \), \( u^n \in 2^{\aleph_0} \), such that still 8° holds, and in addition \( P''_{u^n} \subseteq D^n \) for all \( u^n \in 2^{\aleph_0} \), where \( D^n \) is the \( n \)-th open dense subset of \( P \) coded in \( M \).

Take any \( u^n \in 2^{\aleph_0} \). As \( D^n \) is a dense subset of \( P \), there exists a set \( W_0 \subseteq D^n \), therefore, a non-empty \( \Sigma_1 \) set, such that \( W_0 \subseteq W_{u^n} \). It follows from Lemma 28 that there exists a system of non-empty \( \Sigma_1 \) sets \( W_{u^n} \subseteq W_{u^n} \), \( u^n \in 2^{\aleph_0} \), still satisfying 8°, and such that \( W_{u^n} = W_0 \).

Now take any other \( i_1 \neq i_0 \) in \( 2^{\aleph_0} \). The same construction yields a system of non-empty \( \Sigma_1 \) sets \( W_{u^n} \subseteq W_{u^n} \), \( u^n \in 2^{\aleph_0} \), still satisfying 8°, and such that \( W_{u^n} = W_1 \subseteq W_{u^n} \) is a set in \( D_n \).

Iterating this construction \( 2^{\aleph_0} \) times, we obtain a system of sets \( P''_{u^n} \) satisfying 7° as well as all other conditions in the list 1° – 8°, as required.

\( \square \) (Construction and Case 2 of Theorem 2)

\( \square \) (Theorems 2 and 1)

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